Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Complex numbers
Exercise A, Question 1

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$(5 + 2\text{i}) + (8 + 9\text{i})$

Solution:

$(5 + 8) + \text{i}(2 + 9) = 13 + 11\text{i}$

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Complex numbers
Exercise A, Question 2

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$(4 + 10\text{i}) + (1 - 8\text{i})$

Solution:

$(4 + 1) + \text{i}(10 - 8) = 5 + 2\text{i}$

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Complex numbers
Exercise A, Question 3

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$(7 + 6\text{i}) + (-3 - 5\text{i})$

Solution:

$(7 - 3) + \text{i}(6 - 5) = 4 + \text{i}$

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Complex numbers
Exercise A, Question 4

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$(2 - \text{i}) + (11 + 2\text{i})$

Solution:

$(2 + 11) + \text{i}(-1 + 2) = 13 + \text{i}$

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Complex numbers
Exercise A, Question 5

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$(3 - 7\text{i}) + (-6 + 7\text{i})$

Solution:

$(3 - 6) + \text{i}(-7 + 7) = -3$

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Complex numbers
Exercise A, Question 6

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$(20 + 12\text{i}) - (11 + 3\text{i})$

Solution:

$(20 - 11) + \text{i}(12 - 3) = 9 + 9\text{i}$

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Complex numbers
Exercise A, Question 7

Question:

Simplify, giving your answer in the form $a + b\mathrm{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$(9 + 6\mathrm{i}) - (8 + 10\mathrm{i})$

Solution:

$(9 - 8) + \mathrm{i}(6 - 10) = 1 - 4\mathrm{i}$

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Complex numbers
Exercise A, Question 8

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$(2 - \text{i}) - (-5 + 3\text{i})$

Solution:

$(2 - -5) + \text{i}(-1 - 3) = 7 - 4\text{i}$

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Complex numbers
Exercise A, Question 9

Question:

Simplify, giving your answer in the form $a + b\mathrm{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$(-4 - 6\mathrm{i}) - (-8 - 8\mathrm{i})$

Solution:

$(-4 - -8) + \mathrm{i}(-6 - -8) = 4 + 2\mathrm{i}$

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Complex numbers

Exercise A, Question 10

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$(-1 + 5\text{i}) - (-1 + \text{i})$

Solution:

$(-1 - -1) + \text{i}(5 - 1) = 4\text{i}$

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Complex numbers
Exercise A, Question 11

Question:

Simplify, giving your answer in the form $a + b\mathrm{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$(3 + 4\mathrm{i}) + (4 + 5\mathrm{i}) + (5 + 6\mathrm{i})$

Solution:

$(3 + 4 + 5) + \mathrm{i}(4 + 5 + 6) = 12 + 15\mathrm{i}$

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Complex numbers
Exercise A, Question 12

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$(-2 - 7\text{i}) + (1 + 3\text{i}) - (-12 + \text{i})$

Solution:

$(-2 + 1 - -12) + \text{i}(-7 + 3 - 1) = 11 - 5\text{i}$

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Complex numbers
Exercise A, Question 13

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$(18 + 5\text{i}) - (15 - 2\text{i}) - (3 + 7\text{i})$

Solution:

$(18 - 15 - 3) + \text{i}(5 - -2 - 7) = 0$

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Complex numbers
Exercise A, Question 14

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$2(7 + 2\text{i})$

Solution:

$14 + 4\text{i}$

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Complex numbers
Exercise A, Question 15

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$3(8 - 4\text{i})$

Solution:

$24 - 12\text{i}$

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Complex numbers
Exercise A, Question 16

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$7(1 - 3\text{i})$

Solution:

$7 - 21\text{i}$

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Complex numbers
Exercise A, Question 17

Question:

Simplify, giving your answer in the form $a + b\mathrm{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$2(3 + \mathrm{i}) + 3(2 + \mathrm{i})$

Solution:

$(6 + 2\mathrm{i}) + (6 + 3\mathrm{i}) = (6 + 6) + \mathrm{i}(2 + 3) = 12 + 5\mathrm{i}$

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Complex numbers
Exercise A, Question 18

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$5(4 + 3\text{i}) - 4(-1 + 2\text{i})$

Solution:

$(20 + 15\text{i}) + (4 - 8\text{i}) = (20 + 4) + \text{i}(15 - 8) = 24 + 7\text{i}$

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Complex numbers
Exercise A, Question 19

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$$\left(\frac{1}{2} + \frac{1}{3}\text{i}\right) + \left(\frac{5}{2} + \frac{5}{3}\text{i}\right)$$

Solution:

$$\left(\frac{1}{2} + \frac{5}{2}\right) + \text{i}\left(\frac{1}{3} + \frac{5}{3}\right) = 3 + 2\text{i}$$

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Complex numbers
Exercise A, Question 20

Question:

Simplify, giving your answer in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$(3\sqrt{2} + \text{i}) - (\sqrt{2} - \text{i})$

Solution:

$(3\sqrt{2} - \sqrt{2}) + \text{i}(1 - -1) = 2\sqrt{2} + 2\text{i}$

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Complex numbers
Exercise A, Question 21

Question:

Write in the form *bi*, where $b \in \mathbb{R}$.

$\sqrt{(-9)}$

Solution:

$\sqrt{9}\sqrt{(-1)} = 3\text{i}$

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Complex numbers
Exercise A, Question 22

Question:

Write in the form bi, where $b \in \mathbb{R}$.

$\sqrt{(-49)}$

Solution:

$\sqrt{49}\sqrt{(-1)} = 7\text{i}$

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Complex numbers
Exercise A, Question 23

Question:

Write in the form bi, where $b \in \mathbb{R}$.

$\sqrt{(-121)}$

Solution:

$\sqrt{121}\sqrt{(-1)} = 11\mathrm{i}$

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Complex numbers
Exercise A, Question 24

Question:

Write in the form *bi*, where $b \in \mathbb{R}$.

$\sqrt{(-10000)}$

Solution:

$\sqrt{10000}\sqrt{(-1)} = 100\mathrm{i}$

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Complex numbers
Exercise A, Question 25

Question:

Write in the form *bi*, where $b \in \mathbb{R}$.

$\sqrt{(-225)}$

Solution:

$\sqrt{225}\sqrt{(-1)} = 15\text{i}$

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Complex numbers
Exercise A, Question 26

Question:

Write in the form bi, where $b \in \mathbb{R}$.

$\sqrt{(-5)}$

Solution:

$\sqrt{5}\sqrt{(-1)} = \mathrm{i}\sqrt{5}$

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Complex numbers
Exercise A, Question 27

Question:

Write in the form bi, where $b \in \mathbb{R}$.

$\sqrt{(-12)}$

Solution:

$\sqrt{12}\sqrt{(-1)} = \sqrt{4}\sqrt{3}\sqrt{(-1)} = 2\text{i}\sqrt{3}$

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Complex numbers
Exercise A, Question 28

Question:

Write in the form bi, where $b \in \mathbb{R}$.

$\sqrt{(-45)}$

Solution:

$\sqrt{45}\sqrt{(-1)} = \sqrt{9}\sqrt{5}\sqrt{(-1)} = 3\text{i}\sqrt{5}$

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Complex numbers
Exercise A, Question 29

Question:

Write in the form bi, where $b \in \mathbb{R}$.

$\sqrt{(-200)}$

Solution:

$\sqrt{200}\sqrt{(-1)} = \sqrt{100}\sqrt{2}\sqrt{(-1)} = 10\text{i}\sqrt{2}$

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Complex numbers
Exercise A, Question 30

Question:

Write in the form bi, where $b \in \mathbb{R}$.

$\sqrt{(-147)}$

Solution:

$\sqrt{147}\sqrt{(-1)} = \sqrt{49}\sqrt{3}\sqrt{(-1)} = 7\mathrm{i}\sqrt{3}$

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Complex numbers
Exercise A, Question 31

Question:

Solve these equations.

$x^2 + 2x + 5 = 0$

Solution:

$$\begin{aligned} a &= 1, b = 2, c = 5 \\ x &= \frac{-2 \pm \sqrt{(4-20)}}{2} = \frac{-2 \pm 4\text{i}}{2} \\ x &= -1 \pm 2\text{i} \end{aligned}$$

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Complex numbers
Exercise A, Question 32

Question:

Solve these equations.

$x^2 - 2x + 10 = 0$

Solution:

$a = 1, b = -2, c = 10$

$x = \frac{2 \pm \sqrt{(4 - 40)}}{2} = \frac{2 \pm 6i}{2}$

$x = 1 \pm 3i$

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Complex numbers
Exercise A, Question 33

Question:

Solve these equations.

$x^2 + 4x + 29 = 0$

Solution:

$a = 1, b = 4, c = 29$

$x = \frac{-4 \pm \sqrt{(16 - 116)}}{2} = \frac{-4 \pm 10\text{i}}{2}$

$x = -2 \pm 5\text{i}$

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Complex numbers
Exercise A, Question 34

Question:

Solve these equations.

$x^2 + 10x + 26 = 0$

Solution:

$a = 1, b = 10, c = 26$

$x = \frac{-10 \pm \sqrt{(100 - 104)}}{2} = \frac{-10 \pm 2\text{i}}{2}$

$x = -5 \pm \text{i}$

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Complex numbers
Exercise A, Question 35

Question:

Solve these equations.

$x^2 - 6x + 18 = 0$

Solution:

$$a = 1, b = -6, c = 18$$
$$x = \frac{6 \pm \sqrt{(36 - 72)}}{2} = \frac{6 \pm 6\mathrm{i}}{2}$$
$$x = 3 \pm 3\mathrm{i}$$

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Complex numbers
Exercise A, Question 36

Question:

Solve these equations.

$x^2 + 4x + 7 = 0$

Solution:

$a = 1, b = 4, c = 7$

$x = \frac{-4 \pm \sqrt{(16 - 28)}}{2} = \frac{-4 \pm \mathrm{i}\sqrt{12}}{2} = \frac{-4 \pm 2\mathrm{i}\sqrt{3}}{2}$

$x = -2 \pm \mathrm{i}\sqrt{3}$

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Complex numbers
Exercise A, Question 37

Question:

Solve these equations.

$x^2 - 6x + 11 = 0$

Solution:

$a = 1, b = -6, c = 11$

$x = \frac{6 \pm \sqrt{(36 - 44)}}{2} = \frac{6 \pm i\sqrt{8}}{2} = \frac{6 \pm 2i\sqrt{2}}{2}$

$x = 3 \pm i\sqrt{2}$

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Complex numbers
Exercise A, Question 38

Question:

Solve these equations.

$x^2 - 2x + 25 = 0$

Solution:

$a = 1, b = -2, c = 25$

$x = \frac{2 \pm \sqrt{(4 - 100)}}{2} = \frac{2 \pm i\sqrt{96}}{2} = \frac{2 \pm 4i\sqrt{6}}{2}$

$x = 1 \pm 2i\sqrt{6}$

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Complex numbers
Exercise A, Question 39

Question:

Solve these equations.

$x^2 + 5x + 25 = 0$

Solution:

$a = 1, b = 5, c = 25$

$x = \frac{-5 \pm \sqrt{(25 - 100)}}{2} = \frac{-5 \pm i\sqrt{75}}{2} = \frac{-5 \pm 5i\sqrt{3}}{2}$

$x = \frac{-5}{2} \pm \frac{5i\sqrt{3}}{2}$

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Complex numbers
Exercise A, Question 40

Question:

Solve these equations.

$x^2 + 3x + 5 = 0$

Solution:

$a = 1, b = 3, c = 5$

$x = -3 \pm \frac{\sqrt{(9 - 20)}}{2} = \frac{-3 \pm i\sqrt{11}}{2}$

$x = \frac{-3}{2} \pm \frac{i\sqrt{11}}{2}$

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Complex numbers
Exercise B, Question 1

Question:

Simplify these, giving your answer in the form $a + b\mathrm{i}$.

$(5 + \mathrm{i})(3 + 4\mathrm{i})$

Solution:

$5(3 + 4\mathrm{i}) + \mathrm{i}(3 + 4\mathrm{i})$
$= 15 + 20\mathrm{i} + 3\mathrm{i} + 4\mathrm{i}^2$
$= 15 + 20\mathrm{i} + 3\mathrm{i} - 4$
$= 11 + 23\mathrm{i}$

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Complex numbers
Exercise B, Question 2

Question:

Simplify these, giving your answer in the form $a + b\mathrm{i}$.

$(6 + 3\mathrm{i})(7 + 2\mathrm{i})$

Solution:

$6(7 + 2\mathrm{i}) + 3\mathrm{i}(7 + 2\mathrm{i})$
$= 42 + 12\mathrm{i} + 21\mathrm{i} + 6\mathrm{i}^2$
$= 42 + 12\mathrm{i} + 21\mathrm{i} - 6$
$= 36 + 33\mathrm{i}$

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Complex numbers
Exercise B, Question 3

Question:

Simplify these, giving your answer in the form $a + b\text{i}$.

$(5 - 2\text{i})(1 + 5\text{i})$

Solution:

$5(1 + 5\text{i}) - 2\text{i}(1 + 5\text{i})$
$= 5 + 25\text{i} - 2\text{i} - 10\text{i}^2$
$= 5 + 25\text{i} - 2\text{i} + 10$
$= 15 + 23\text{i}$

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Complex numbers
Exercise B, Question 4

Question:

Simplify these, giving your answer in the form $a + b\text{i}$.

$(13 - 3\text{i})(2 - 8\text{i})$

Solution:

$13(2 - 8\text{i}) - 3\text{i}(2 - 8\text{i})$
$= 26 - 104\text{i} - 6\text{i} + 24\text{i}^2$
$= 26 - 104\text{i} - 6\text{i} - 24$
$= 2 - 110\text{i}$

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Complex numbers
Exercise B, Question 5

Question:

Simplify these, giving your answer in the form $a + b\mathrm{i}$.

$(-3 - \mathrm{i})(4 + 7\mathrm{i})$

Solution:

$-3(4 + 7\mathrm{i}) - \mathrm{i}(4 + 7\mathrm{i})$
$= -12 - 21\mathrm{i} - 4\mathrm{i} - 7\mathrm{i}^2$
$= -12 - 21\mathrm{i} - 4\mathrm{i} + 7$
$= -5 - 25\mathrm{i}$

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Complex numbers
Exercise B, Question 6

Question:

Simplify these, giving your answer in the form $a + b\text{i}$.

$(8 + 5\text{i})^2$

Solution:

$(8 + 5\text{i})(8 + 5\text{i}) = 8(8 + 5\text{i}) + 5\text{i}(8 + 5\text{i})$
$= 64 + 40\text{i} + 40\text{i} + 25\text{i}^2$
$= 64 + 40\text{i} + 40\text{i} - 25$
$= 39 + 80\text{i}$

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Complex numbers
Exercise B, Question 7

Question:

Simplify these, giving your answer in the form $a + b\text{i}$.

$(2 - 9\text{i})^2$

Solution:

$(2 - 9\text{i})(2 - 9\text{i}) = 2(2 - 9\text{i}) - 9\text{i}(2 - 9\text{i})$
$= 4 - 18\text{i} - 18\text{i} + 81\text{i}^2$
$= 4 - 18\text{i} - 18\text{i} - 81$
$= -77 - 36\text{i}$

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Complex numbers
Exercise B, Question 8

Question:

Simplify these, giving your answer in the form $a + b\text{i}$.

$(1 + \text{i})(2 + \text{i})(3 + \text{i})$

Solution:

$(2 + \text{i})(3 + \text{i}) = 2(3 + \text{i}) + \text{i}(3 + \text{i})$
$= 6 + 2\text{i} + 3\text{i} + \text{i}^2$
$= 6 + 2\text{i} + 3\text{i} - 1$
$= 5 + 5\text{i}$
$(1 + \text{i})(5 + 5\text{i}) = 1(5 + 5\text{i}) + \text{i}(5 + 5\text{i})$
$= 5 + 5\text{i} + 5\text{i} + 5\text{i}^2$
$= 5 + 5\text{i} + 5\text{i} - 5$
$= 10\text{i}$

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Complex numbers
Exercise B, Question 9

Question:

Simplify these, giving your answer in the form $a + b\mathrm{i}$.

$(3 - 2\mathrm{i})(5 + \mathrm{i})(4 - 2\mathrm{i})$

Solution:

$(5 + \mathrm{i})(4 - 2\mathrm{i}) = 5(4 - 2\mathrm{i}) + \mathrm{i}(4 - 2\mathrm{i})$
$= 20 - 10\mathrm{i} + 4\mathrm{i} - 2\mathrm{i}^2$
$= 20 - 10\mathrm{i} + 4\mathrm{i} + 2$
$= 22 - 6\mathrm{i}$
$(3 - 2\mathrm{i})(22 - 6\mathrm{i}) = 3(22 - 6\mathrm{i}) - 2\mathrm{i}(22 - 6\mathrm{i})$
$= 66 - 18\mathrm{i} - 44\mathrm{i} + 12\mathrm{i}^2$
$= 66 - 18\mathrm{i} - 44\mathrm{i} - 12$
$= 54 - 62\mathrm{i}$

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Complex numbers
Exercise B, Question 10

Question:

Simplify these, giving your answer in the form $a + b\text{i}$.

$(2 + 3\text{i})^3$

Solution:

$(2 + 3\text{i})^2 = (2 + 3\text{i})(2 + 3\text{i})$
$= 2(2 + 3\text{i}) + 3\text{i}(2 + 3\text{i})$
$= 4 + 6\text{i} + 6\text{i} + 9\text{i}^2$
$= 4 + 6\text{i} + 6\text{i} - 9$
$= -5 + 12\text{i}$
$(2 + 3\text{i})^3 = (2 + 3\text{i})(-5 + 12\text{i})$
$= 2(-5 + 12\text{i}) + 3\text{i}(-5 + 12\text{i})$
$= -10 + 24\text{i} - 15\text{i} + 36\text{i}^2$
$= -10 + 24\text{i} - 15\text{i} - 36$
$= -46 + 9\text{i}$

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Complex numbers
Exercise B, Question 11

Question:

Simplify

i^6

Solution:

$i \times i \times i \times i \times i \times i$
$= i^2 \times i^2 \times i^2 = -1 \times -1 \times -1 = -1$

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Complex numbers
Exercise B, Question 12

Question:

Simplify

$(3i)^4$

Solution:

$3i \times 3i \times 3i \times 3i$
$= 81(i \times i \times i \times i) = 81(i^2 \times i^2)$
$= 81(-1 \times -1) = 81$

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Complex numbers
Exercise B, Question 13

Question:

Simplify

$i^5 + i$

Solution:

$(i \times i \times i \times i \times i) + i$
$= (i^2 \times i^2 \times i) + i = (-1 \times -1 \times i) + i$
$= i + i = 2i$

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Complex numbers
Exercise B, Question 14

Question:

Simplify

$(4i)^3 - 4i^3$

Solution:

$(4i)^3 = 4i \times 4i \times 4i = 64(i \times i \times i)$
$= 64(-1 \times i) = -64i$
$4i^3 = 4(i \times i \times i) = 4(-1 \times i) = -4i$
$(4i)^3 - 4i^3 = -64i - (-4i)$
$= -64i + 4i$
$= -60i$

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Complex numbers
Exercise B, Question 15

Question:

Simplify

$(1+i)^8$

Solution:

$(1+i)^8$

$$= 1^8 + 8.1^7 i + 28.1^6 i^2 + 56.1^5 i^3 + 70.1^4 i^4 + 56.1^3 i^5 + 28.1^2 i^6 + 8.1 i^7 + i^8$$
$$= 1 + 8i + 28i^2 + 56i^3 + 70i^4 + 56i^5 + 28i^6 + 8i^7 + i^8$$

$$\begin{aligned}
i^2 &= -1 \\
i^3 &= i^2 \times i = -i \\
i^4 &= i^2 \times i^2 = 1 \\
i^5 &= i^2 \times i^2 \times i = i \\
i^6 &= i^2 \times i^2 \times i^2 = -1 \\
i^7 &= i^2 \times i^2 \times i^2 \times i = -i \\
i^8 &= i^2 \times i^2 \times i^2 \times i^2 = 1 \\
(1+i)^8 &= 1 + 8i - 28 - 56i + 70 + 56i - 28 - 8i + 1 \\
&= 16
\end{aligned}$$

Note also that $(1+i)^2 = (1+i)(1+i)$

$= 1 + 2i + i^2 = 2i$

So $(1+i)^8 = (2i)^4 = 16i^4 = 16$

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Complex numbers
Exercise C, Question 1

Question:

Write down the complex conjugate z^* for

a $z = 8 + 2\text{i}$

b $z = 6 - 5\text{i}$

c $z = \frac{2}{3} - \frac{1}{2}\text{i}$

d $z = \sqrt{5} + \text{i}\sqrt{10}$

Solution:

a $z^* = 8 - 2\text{i}$

b $z^* = 6 + 5\text{i}$

c $z^* = \frac{2}{3} + \frac{1}{2}\text{i}$

d $z^* = \sqrt{5} - \text{i}\sqrt{10}$

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Complex numbers
Exercise C, Question 2

Question:

Find $z + z^*$ and zz^* for

a $z = 6 - 3\text{i}$

b $z = 10 + 5\text{i}$

c $z = \frac{3}{4} + \frac{1}{4}\text{i}$

d $z = \sqrt{5} - 3\text{i}\sqrt{5}$

Solution:

a

$$\begin{aligned} z + z^* &= (6 - 3\text{i}) + (6 + 3\text{i}) = 12 \\ zz^* &= (6 - 3\text{i})(6 + 3\text{i}) \\ &= 6(6 + 3\text{i}) - 3\text{i}(6 + 3\text{i}) \\ &= 36 + 18\text{i} - 18\text{i} - 9\text{i}^2 = 45 \end{aligned}$$

b

$$\begin{aligned} z + z^* &= (10 + 5\text{i}) + (10 - 5\text{i}) = 20 \\ zz^* &= (10 + 5\text{i})(10 - 5\text{i}) \\ &= 10(10 - 5\text{i}) + 5\text{i}(10 - 5\text{i}) \\ &= 100 - 50\text{i} + 50\text{i} - 25\text{i}^2 = 125 \end{aligned}$$

c

$$\begin{aligned} z + z^* &= \left(\frac{3}{4} + \frac{1}{4}\text{i}\right) + \left(\frac{3}{4} - \frac{1}{4}\text{i}\right) = \frac{3}{2} \\ zz^* &= \left(\frac{3}{4} + \frac{1}{4}\text{i}\right)\left(\frac{3}{4} - \frac{1}{4}\text{i}\right) \\ &= \frac{3}{4}\left(\frac{3}{4} - \frac{1}{4}\text{i}\right) + \frac{1}{4}\text{i}\left(\frac{3}{4} - \frac{1}{4}\text{i}\right) \\ &= \frac{9}{16} - \frac{3}{16}\text{i} + \frac{3}{16}\text{i} - \frac{1}{16}\text{i}^2 \\ &= \frac{10}{16} = \frac{5}{8} \end{aligned}$$

d

$$\begin{aligned} z + z^* &= (\sqrt{5} - 3\text{i}\sqrt{5}) + (\sqrt{5} + 3\text{i}\sqrt{5}) = 2\sqrt{5} \\ zz^* &= (\sqrt{5} - 3\text{i}\sqrt{5})(\sqrt{5} + 3\text{i}\sqrt{5}) \\ &= \sqrt{5}(\sqrt{5} + 3\text{i}\sqrt{5}) - 3\text{i}\sqrt{5}(\sqrt{5} + 3\text{i}\sqrt{5}) \\ &= 5 + 15\text{i} - 15\text{i} - 45\text{i}^2 \\ &= 50 \end{aligned}$$

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Complex numbers
Exercise C, Question 3

Question:

Find these in the form $a + b\text{i}$.

$(25 - 10\text{i}) \div (1 - 2\text{i})$

Solution:

$$\begin{aligned}
\frac{25 - 10\text{i}}{1 - 2\text{i}} &= \frac{(25 - 10\text{i})(1 + 2\text{i})}{(1 - 2\text{i})(1 + 2\text{i})} \\
(25 - 10\text{i})(1 + 2\text{i}) &= 25(1 + 2\text{i}) - 10\text{i}(1 + 2\text{i}) \\
&= 25 + 50\text{i} - 10\text{i} - 20\text{i}^2 \\
&= 45 + 40\text{i} \\
(1 - 2\text{i})(1 + 2\text{i}) &= 1(1 + 2\text{i}) - 2\text{i}(1 + 2\text{i}) \\
&= 1 + 2\text{i} - 2\text{i} - 4\text{i}^2 \\
&= 5 \\
\frac{45 + 40\text{i}}{5} &= 9 + 8\text{i}
\end{aligned}$$

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Complex numbers
Exercise C, Question 4

Question:

Find these in the form $a + b\text{i}$.

$(6 + \text{i}) \div (3 + 4\text{i})$

Solution:

$$\frac{6+\text{i}}{3+4\text{i}} = \frac{(6+\text{i})(3-4\text{i})}{(3+4\text{i})(3-4\text{i})}$$

$$\begin{aligned}(6+\text{i})(3-4\text{i}) &= 6(3-4\text{i})+\text{i}(3-4\text{i}) \\ &= 18-24\text{i}+3\text{i}-4\text{i}^2 \\ &= 22-21\text{i} \\ (3+4\text{i})(3-4\text{i}) &= 3(3-4\text{i})+4\text{i}(3-4\text{i}) \\ &= 9-12\text{i}+12\text{i}-16\text{i}^2 \\ &= 25 \\ \frac{22-21\text{i}}{25} &= \frac{22}{25}-\frac{21}{25}\text{i}\end{aligned}$$

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Complex numbers
Exercise C, Question 5

Question:

Find these in the form $a + b\text{i}$.

$(11 + 4\text{i}) \div (3 + \text{i})$

Solution:

$$\begin{aligned}
\frac{11+4\text{i}}{3+\text{i}} &= \frac{(11+4\text{i})(3-\text{i})}{(3+\text{i})(3-\text{i})} \\
(11+4\text{i})(3-\text{i}) &= 11(3-\text{i}) + 4\text{i}(3-\text{i}) \\
&= 33 - 11\text{i} + 12\text{i} - 4\text{i}^2 \\
&= 37 + \text{i} \\
(3+\text{i})(3-\text{i}) &= 3(3-\text{i}) + \text{i}(3-\text{i}) \\
&= 9 - 3\text{i} + 3\text{i} - \text{i}^2 \\
&= 10 \\
\frac{37+\text{i}}{10} &= \frac{37}{10} + \frac{1}{10}\text{i}
\end{aligned}$$

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Complex numbers
Exercise C, Question 6

Question:

Find these in the form $a + b\mathrm{i}$.

$$\frac{1+\mathrm{i}}{2+\mathrm{i}}$$

Solution:

$$\begin{aligned}
\frac{1+\mathrm{i}}{2+\mathrm{i}} &= \frac{(1+\mathrm{i})(2-\mathrm{i})}{(2+\mathrm{i})(2-\mathrm{i})} \\
(1+\mathrm{i})(2-\mathrm{i}) &= 1(2-\mathrm{i})+\mathrm{i}(2-\mathrm{i}) \\
&= 2-\mathrm{i}+2\mathrm{i}-\mathrm{i}^2 \\
&= 3+\mathrm{i} \\
(2+\mathrm{i})(2-\mathrm{i}) &= 2(2-\mathrm{i})+\mathrm{i}(2-\mathrm{i}) \\
&= 4-2\mathrm{i}+2\mathrm{i}-\mathrm{i}^2 \\
&= 5 \\
\frac{3+\mathrm{i}}{5} &= \frac{3}{5}+\frac{1}{5}\mathrm{i}
\end{aligned}$$

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Complex numbers
Exercise C, Question 7

Question:

Find these in the form $a + b\text{i}$.

$$\frac{3-5\text{i}}{1+3\text{i}}$$

Solution:

$$\frac{3-5\text{i}}{1+3\text{i}} = \frac{(3-5\text{i})(1-3\text{i})}{(1+3\text{i})(1-3\text{i})}$$

$$\begin{aligned}(3-5\text{i})(1-3\text{i}) &= 3(1-3\text{i})-5\text{i}(1-3\text{i})\\ &= 3-9\text{i}-5\text{i}+15\text{i}^2\\ &= -12-14\text{i}\\ (1+3\text{i})(1-3\text{i}) &= 1(1-3\text{i})+3\text{i}(1-3\text{i})\\ &= 1-3\text{i}+3\text{i}-9\text{i}^2\\ &= 10\\ \frac{-12-14\text{i}}{10} &= -\frac{6}{5}-\frac{7}{5}\text{i}\end{aligned}$$

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Complex numbers
Exercise C, Question 8

Question:

Find these in the form $a + b\text{i}$.

$$\frac{3+5\text{i}}{6-8\text{i}}$$

Solution:

$$\frac{3+5\text{i}}{6-8\text{i}} = \frac{(3+5\text{i})(6+8\text{i})}{(6-8\text{i})(6+8\text{i})}$$

$$\begin{aligned}(3+5\text{i})(6+8\text{i}) &= 3(6+8\text{i})+5\text{i}(6+8\text{i})\\ &= 18+24\text{i}+30\text{i}+40\text{i}^2\\ &= -22+54\text{i}\\ (6-8\text{i})(6+8\text{i}) &= 6(6+8\text{i})-8\text{i}(6+8\text{i})\\ &= 36+48\text{i}-48\text{i}-64\text{i}^2\\ &= 100\\ \frac{-22+54\text{i}}{100} &= \frac{-11}{50}+\frac{27}{50}\text{i}\end{aligned}$$

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Complex numbers

Exercise C, Question 9

Question:

Find these in the form $a + b\text{i}$.

$\frac{28 - 3\text{i}}{1 - \text{i}}$

Solution:

$$\begin{aligned}
\frac{28 - 3\text{i}}{1 - \text{i}} &= \frac{(28 - 3\text{i})(1 + \text{i})}{(1 - \text{i})(1 + \text{i})} \\
(28 - 3\text{i})(1 + \text{i}) &= 28(1 + \text{i}) - 3\text{i}(1 + \text{i}) \\
&= 28 + 28\text{i} - 3\text{i} - 3\text{i}^2 \\
&= 31 + 25\text{i} \\
(1 - \text{i})(1 + \text{i}) &= 1(1 + \text{i}) - \text{i}(1 + \text{i}) \\
&= 1 + \text{i} - \text{i} - \text{i}^2 \\
&= 2 \\
\frac{31 + 25\text{i}}{2} &= \frac{31}{2} + \frac{25}{2}\text{i}
\end{aligned}$$

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Complex numbers
Exercise C, Question 10

Question:

Find these in the form $a + b\text{i}$.

$$\frac{2+\text{i}}{1+4\text{i}}$$

Solution:

$$\begin{aligned}
\frac{2+\text{i}}{1+4\text{i}} &= \frac{(2+\text{i})(1-4\text{i})}{(1+4\text{i})(1-4\text{i})} \\
(2+\text{i})(1-4\text{i}) &= 2(1-4\text{i}) + \text{i}(1-4\text{i}) \\
&= 2 - 8\text{i} + \text{i} - 4\text{i}^2 \\
&= 6 - 7\text{i} \\
(1+4\text{i})(1-4\text{i}) &= 1(1-4\text{i}) + 4\text{i}(1-4\text{i}) \\
&= 1 - 4\text{i} + 4\text{i} - 16\text{i}^2 \\
&= 17 \\
\frac{6-7\text{i}}{17} &= \frac{6}{17} - \frac{7}{17}\text{i}
\end{aligned}$$

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Complex numbers
Exercise C, Question 11

Question:

Find these in the form $a + b\text{i}$.

$$\frac{(3-4\text{i})^2}{1+\text{i}}$$

Solution:

$$\begin{aligned}
(3-4\text{i})^2 &= (3-4\text{i})(3-4\text{i}) \\
&= 3(3-4\text{i}) - 4\text{i}(3-4\text{i}) \\
&= 9 - 12\text{i} - 12\text{i} + 16\text{i}^2 \\
&= -7 - 24\text{i} \\
\frac{-7-24\text{i}}{1+\text{i}} &= \frac{(-7-24\text{i})(1-\text{i})}{(1+\text{i})(1-\text{i})} \\
(-7-24\text{i})(1-\text{i}) &= -7(1-\text{i}) - 24\text{i}(1-\text{i}) \\
&= -7 + 7\text{i} - 24\text{i} + 24\text{i}^2 \\
&= -31 - 17\text{i} \\
(1+\text{i})(1-\text{i}) &= 1(1-\text{i}) + \text{i}(1-\text{i}) \\
&= 1 - \text{i} + \text{i} - \text{i}^2 \\
&= 2 \\
\frac{-31-17\text{i}}{2} &= \frac{-31}{2} - \frac{17}{2}\text{i}
\end{aligned}$$

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Complex numbers
Exercise C, Question 12

Question:

Given that $z_1 = 1 + \mathrm{i}, z_2 = 2 + \mathrm{i}$ and $z_3 = 3 + \mathrm{i}$, find the following in the form $a + b\mathrm{i}$.

$$\frac{z_1 z_2}{z_3}$$

Solution:

$$\begin{aligned}
z_1 z_2 &= (1+\mathrm{i})(2+\mathrm{i}) \\
&= 1(2+\mathrm{i}) + \mathrm{i}(2+\mathrm{i}) \\
&= 2 + \mathrm{i} + 2\mathrm{i} + \mathrm{i}^2 \\
&= 1 + 3\mathrm{i} \\
\frac{z_1 z_2}{z_3} &= \frac{1+3\mathrm{i}}{3+\mathrm{i}} = \frac{(1+3\mathrm{i})(3-\mathrm{i})}{(3+\mathrm{i})(3-\mathrm{i})} \\
(1+3\mathrm{i})(3-\mathrm{i}) &= 1(3-\mathrm{i}) + 3\mathrm{i}(3-\mathrm{i}) \\
&= 3 - \mathrm{i} + 9\mathrm{i} - 3\mathrm{i}^2 \\
&= 6 + 8\mathrm{i} \\
(3+\mathrm{i})(3-\mathrm{i}) &= 3(3-\mathrm{i}) + \mathrm{i}(3-\mathrm{i}) \\
&= 9 - 3\mathrm{i} + 3\mathrm{i} - \mathrm{i}^2 \\
&= 10 \\
\frac{6+8\mathrm{i}}{10} &= \frac{3}{5} + \frac{4}{5}\mathrm{i}
\end{aligned}$$

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Complex numbers
Exercise C, Question 13

Question:

Given that $z_1 = 1 + \mathrm{i}, z_2 = 2 + \mathrm{i}$ and $z_3 = 3 + \mathrm{i}$, find the following in the form $a + b\mathrm{i}$.

$$\frac{(z_2)^2}{z_1}$$

Solution:

$$\begin{aligned}
(z_2)^2 &= (2+\mathrm{i})(2+\mathrm{i}) \\
&= 2(2+\mathrm{i}) + \mathrm{i}(2+\mathrm{i}) \\
&= 4 + 2\mathrm{i} + 2\mathrm{i} + \mathrm{i}^2 \\
&= 3 + 4\mathrm{i} \\
\frac{(z_2)^2}{z_1} &= \frac{3+4\mathrm{i}}{1+\mathrm{i}} = \frac{(3+4\mathrm{i})(1-\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})} \\
(3+4\mathrm{i})(1-\mathrm{i}) &= 3(1-\mathrm{i}) + 4\mathrm{i}(1-\mathrm{i}) \\
&= 3 - 3\mathrm{i} + 4\mathrm{i} - 4\mathrm{i}^2 \\
&= 7 + \mathrm{i} \\
(1+\mathrm{i})(1-\mathrm{i}) &= 1(1-\mathrm{i}) + \mathrm{i}(1-\mathrm{i}) \\
&= 1 - \mathrm{i} + \mathrm{i} - \mathrm{i}^2 \\
&= 2 \\
\frac{7+\mathrm{i}}{2} &= \frac{7}{2} + \frac{1}{2}\mathrm{i}
\end{aligned}$$

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Complex numbers
Exercise C, Question 14

Question:

Given that $z_1 = 1 + \mathrm{i}, z_2 = 2 + \mathrm{i}$ and $z_3 = 3 + \mathrm{i}$, find the following in the form $a + b\mathrm{i}$.

$$\frac{2z_1 + 5z_3}{z_2}$$

Solution:

$$\begin{aligned}
2z_1 + 5z_3 &= 2(1 + \mathrm{i}) + 5(3 + \mathrm{i}) \\
&= 2 + 2\mathrm{i} + 15 + 5\mathrm{i} \\
&= 17 + 7\mathrm{i} \\
\frac{2z_1 + 5z_3}{z_2} &= \frac{17 + 7\mathrm{i}}{2 + \mathrm{i}} = \frac{(17 + 7\mathrm{i})(2 - \mathrm{i})}{(2 + \mathrm{i})(2 - \mathrm{i})} \\
(17 + 7\mathrm{i})(2 - \mathrm{i}) &= 17(2 - \mathrm{i}) + 7\mathrm{i}(2 - \mathrm{i}) \\
&= 34 - 17\mathrm{i} + 14\mathrm{i} - 7\mathrm{i}^2 \\
&= 41 - 3\mathrm{i} \\
(2 + \mathrm{i})(2 - \mathrm{i}) &= 2(2 - \mathrm{i}) + \mathrm{i}(2 - \mathrm{i}) \\
&= 4 - 2\mathrm{i} + 2\mathrm{i} - \mathrm{i}^2 \\
&= 5 \\
\frac{41 - 3\mathrm{i}}{5} &= \frac{41}{5} - \frac{3}{5}\mathrm{i}
\end{aligned}$$

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Complex numbers
Exercise C, Question 15

Question:

Given that $\frac{5+2\text{i}}{z} = 2 - \text{i}$, find z in the form $a + b\text{i}$.

Solution:

$$\frac{5+2\text{i}}{z} = 2 - \text{i}$$

$$z = \frac{5+2\text{i}}{2-\text{i}} = \frac{(5+2\text{i})(2+\text{i})}{(2-\text{i})(2+\text{i})}$$

$$(5+2\text{i})(2+\text{i}) = 5(2+\text{i}) + 2\text{i}(2+\text{i})$$

$$= 10 + 5\text{i} + 4\text{i} + 2\text{i}^2$$

$$= 8 + 9\text{i}$$

$$(2-\text{i})(2+\text{i}) = 2(2+\text{i}) - \text{i}(2+\text{i})$$

$$= 4 + 2\text{i} - 2\text{i} - \text{i}^2$$

$$= 5$$

$$z = \frac{8+9\text{i}}{5} = \frac{8}{5} + \frac{9}{5}\text{i}$$

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Complex numbers
Exercise C, Question 16

Question:

Simplify $\frac{6+8\text{i}}{1+\text{i}}+\frac{6+8\text{i}}{1-\text{i}}$, giving your answer in the form $a+b\text{i}$.

Solution:

$$\frac{6+8\text{i}}{1+\text{i}}+\frac{6+8\text{i}}{1-\text{i}}$$
$$=\frac{(6+8\text{i})(1-\text{i})+(6+8\text{i})(1+\text{i})}{(1+\text{i})(1-\text{i})}$$
$$=\frac{6(1-\text{i})+8\text{i}(1-\text{i})+6(1+\text{i})+8\text{i}(1+\text{i})}{1(1-\text{i})+\text{i}(1-\text{i})}$$
$$=\frac{6-6\text{i}+8\text{i}-8\text{i}^2+6+6\text{i}+8\text{i}+8\text{i}^2}{1-\text{i}+\text{i}-\text{i}^2}$$
$$=\frac{12+16\text{i}}{2}=6+8\text{i}$$

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Complex numbers
Exercise C, Question 17

Question:

The roots of the quadratic equation $x^2+2x+26=0$ are α and β. Find

a α and β

b $\alpha+\beta$

c $\alpha\beta$

Solution:

$x^2+2x+26=0$
$a=1\ ,b=2\ ,c=26$
$x=\dfrac{-2\pm\sqrt{(4-104)}}{2}=\dfrac{-2\pm 10\mathrm{i}}{2}$

a $\alpha=-1+5\mathrm{i}, \beta=-1-5\mathrm{i}$ or vice versa

b $\alpha+\beta=(-1+5\mathrm{i})+(-1-5\mathrm{i})=-2$

c

$$\begin{aligned}\alpha\beta &= (-1+5\mathrm{i})(-1-5\mathrm{i})\\ &= -1(-1-5\mathrm{i})+5\mathrm{i}(-1-5\mathrm{i})\\ &= 1+5\mathrm{i}-5\mathrm{i}-25\mathrm{i}^2=26\end{aligned}$$

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Complex numbers
Exercise C, Question 18

Question:

The roots of the quadratic equation $x^2 - 8x + 25 = 0$ are α and β. Find

a α and β

b $\alpha + \beta$

c $\alpha\beta$

Solution:

$x^2 - 8x + 25 = 0$
$a = 1 \;, b = -8 \;, c = 25$
$x = \dfrac{8 \pm \sqrt{(64 - 100)}}{2} = \dfrac{8 \pm 6i}{2}$

(a) $\alpha = 4 + 3i, \beta = 4 - 3i$ or vice versa

(b) $\alpha + \beta = (4 + 3i) + (4 - 3i) = 8$

(c) $\alpha\beta \;\; = (4 + 3i)(4 - 3i)$

$= 4(4 - 3i) + 3i(4 - 3i)$
$= 16 - 12i + 12i - 9i^2 = 25$

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Complex numbers
Exercise C, Question 19

Question:

Find the quadratic equation that has roots $2+3\text{i}$ and $2-3\text{i}$.

Solution:

If roots are α and β, the equation is

$$(x-\alpha)(x-\beta) \quad = x^2-(\alpha+\beta)x+\alpha\beta=0$$

$$\begin{aligned}\alpha+\beta &= (2+3\text{i})+(2-3\text{i})=4\\ \alpha\beta &= (2+3\text{i})(2-3\text{i})\\ &= 2(2-3\text{i})+3\text{i}(2-3\text{i})\\ &= 4-6\text{i}+6\text{i}-9\text{i}^2=13\end{aligned}$$

Equation is $x^2-4x+13=0$

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Complex numbers
Exercise C, Question 20

Question:

Find the quadratic equation that has roots $-5 + 4\text{i}$ and $-5 - 4\text{i}$.

Solution:

If roots are α and β, the equation is

$$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

$$\begin{aligned}
\alpha+\beta &= (-5+4\text{i}) + (-5-4\text{i}) = -10 \\
\alpha\beta &= (-5+4\text{i})(-5-4\text{i}) \\
&= -5(-5-4\text{i}) + 4\text{i}(-5-4\text{i}) \\
&= 25 + 20\text{i} - 20\text{i} - 16\text{i}^2 \\
&= 41
\end{aligned}$$

Equation is $x^2 + 10x + 41 = 0$

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Complex numbers
Exercise D, Question 1

Question:

Show these numbers on an Argand diagram.

a $7+2i$

b $5-4i$

c $-6-i$

d $-2+5i$

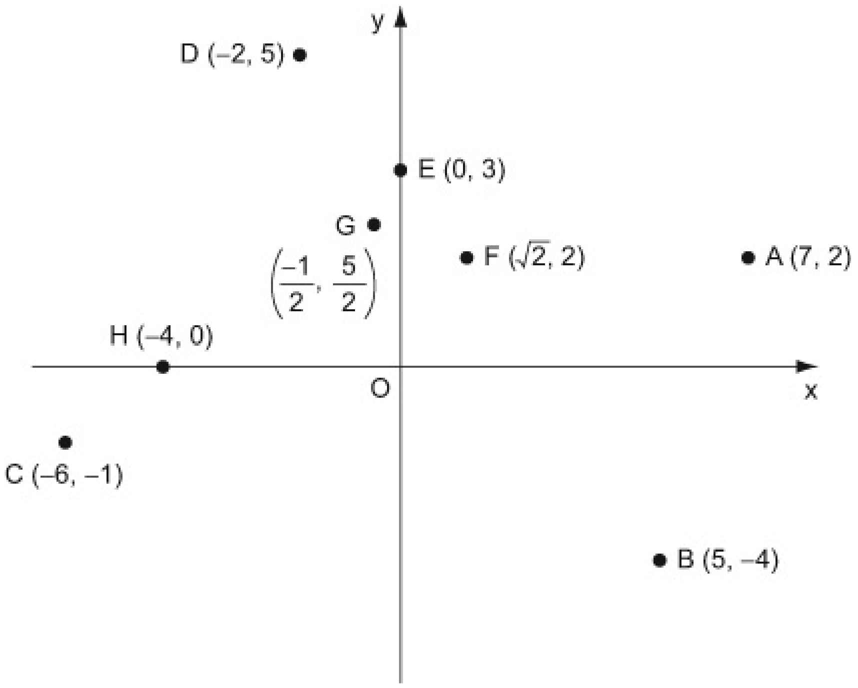
e $3i$

f $\sqrt{2}+2i$

g $-\frac{1}{2}+\frac{5}{2}i$

h -4

Solution:



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Complex numbers
Exercise D, Question 2

Question:

Given that $z_1 = -1 - \mathrm{i}$, $z_2 = -5 + 10\mathrm{i}$ and $z_3 = 3 - 4\mathrm{i}$,

a find z_1z_2, z_1z_3 and $\frac{z_2}{z_3}$ in the form $a + \mathrm{i}b$.

b show $z_1, z_2, z_3, z_1z_2, z_1z_3$ and $\frac{z_2}{z_3}$ on an Argand diagram.

Solution:

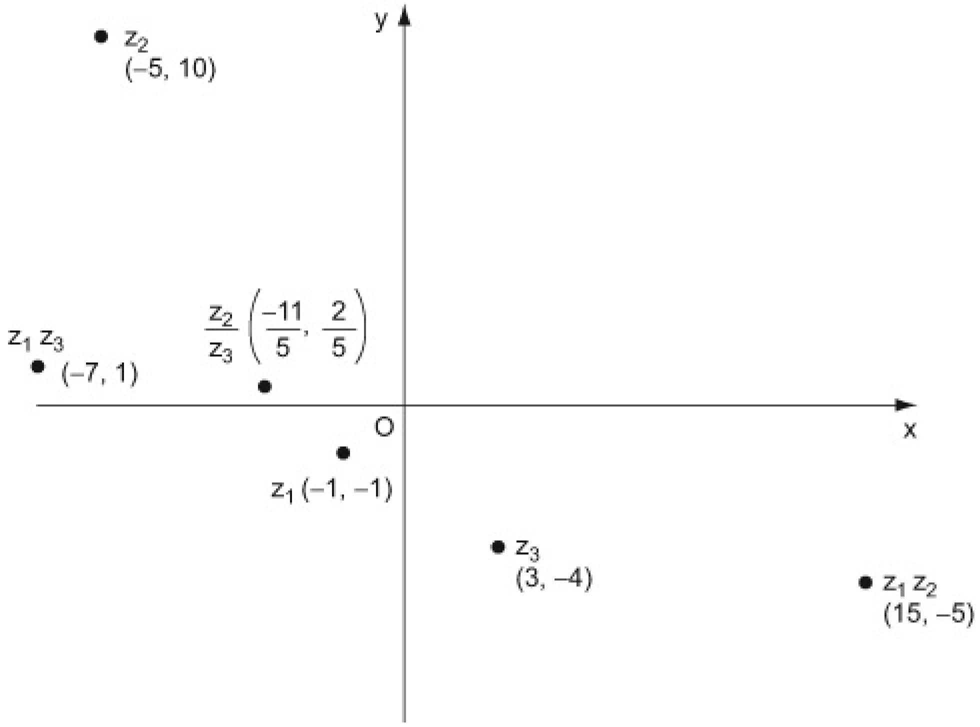
a $z_1z_2 = (-1 - \mathrm{i})(-5 + 10\mathrm{i})$

$$= -1(-5 + 10\mathrm{i}) - \mathrm{i}(-5 + 10\mathrm{i})$$
$$= 5 - 10\mathrm{i} + 5\mathrm{i} - 10\mathrm{i}^2$$
$$= 15 - 5\mathrm{i}$$

$$z_1z_3 = (-1 - \mathrm{i})(3 - 4\mathrm{i})$$
$$= -1(3 - 4\mathrm{i}) - \mathrm{i}(3 - 4\mathrm{i})$$
$$= -3 + 4\mathrm{i} - 3\mathrm{i} + 4\mathrm{i}^2$$
$$= -7 + \mathrm{i}$$

$$\frac{z_2}{z_3} = \frac{-5 + 10\mathrm{i}}{3 - 4\mathrm{i}} = \frac{(-5 + 10\mathrm{i})(3 + 4\mathrm{i})}{(3 - 4\mathrm{i})(3 + 4\mathrm{i})}$$
$$= \frac{-5(3 + 4\mathrm{i}) + 10\mathrm{i}(3 + 4\mathrm{i})}{3(3 + 4\mathrm{i}) - 4\mathrm{i}(3 + 4\mathrm{i})}$$
$$= \frac{-15 - 20\mathrm{i} + 30\mathrm{i} + 40\mathrm{i}^2}{9 + 12\mathrm{i} - 12\mathrm{i} - 16\mathrm{i}^2}$$
$$= \frac{-55 + 10\mathrm{i}}{25} = \frac{-11}{5} + \frac{2}{5}\mathrm{i}$$

b


y
z_2
(–5, 10)
$z_1 z_3$
(–7, 1)
$\frac{z_2}{z_3}\left(\frac{-11}{5}, \frac{2}{5}\right)$
O
x
z_1 (–1, –1)
z_3
(3, –4)
$z_1 z_2$
(15, –5)

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Complex numbers
Exercise D, Question 3

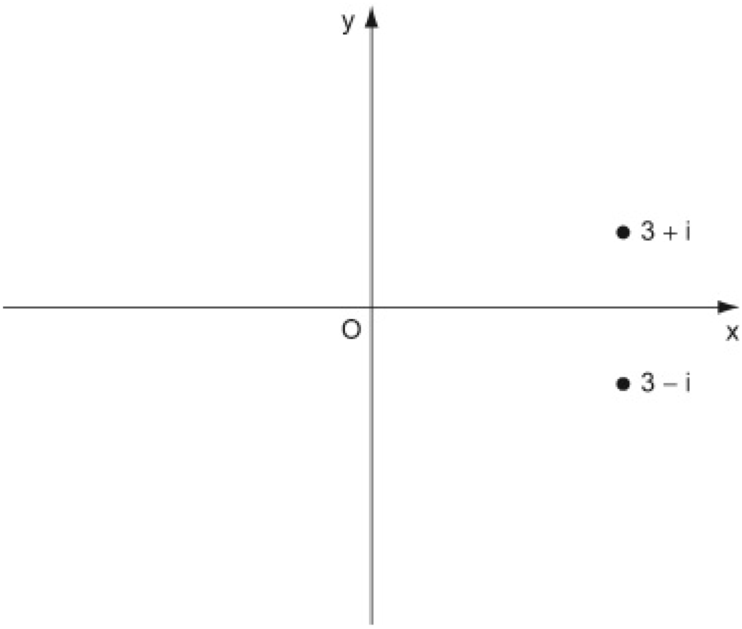
Question:

Show the roots of the equation $x^2 - 6x + 10 = 0$ on an Argand diagram.

Solution:

$x^2 - 6x + 10 = 0$
$a = 1, b = -6, c = 10$
$x = \frac{6 \pm \sqrt{(36 - 40)}}{2} = \frac{6 \pm 2i}{2}$

Roots are $3 + i$ and $3 - i$



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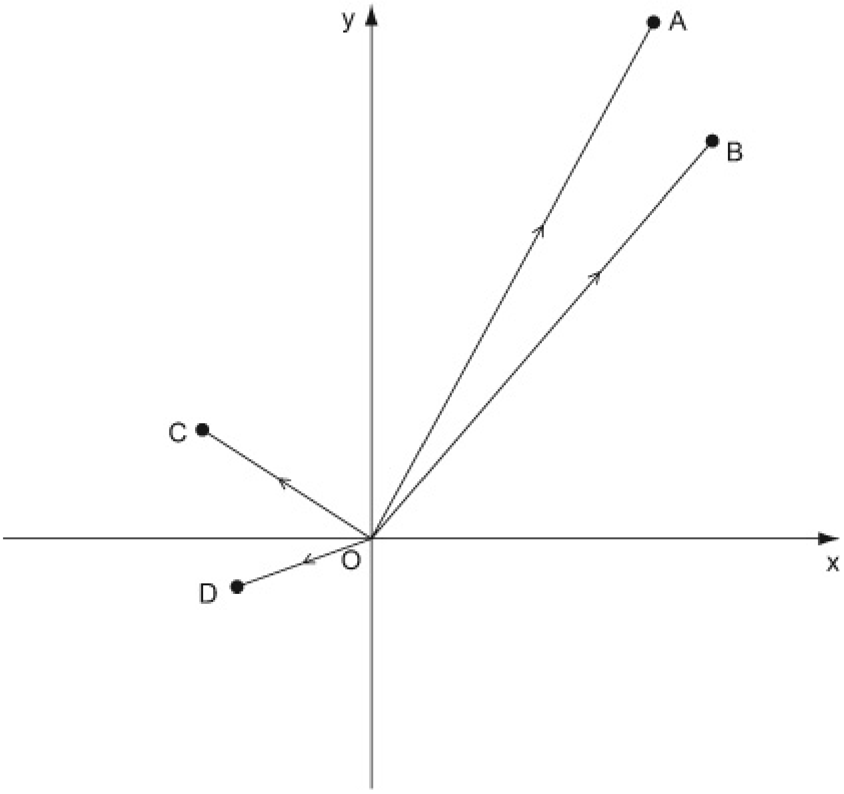
Complex numbers
Exercise D, Question 4

Question:

The complex numbers $z_1 = 5 + 12\text{i}, z_2 = 6 + 10\text{i}, z_3 = -4 + 2\text{i}$ and $z_4 = -3 - \text{i}$ are represented by the vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively on an Argand diagram. Draw the diagram and calculate $|\overrightarrow{OA}|$, $|\overrightarrow{OB}|$, $|\overrightarrow{OC}|$ and $|\overrightarrow{OD}|$.

Solution:



$|\overrightarrow{OA}| = \sqrt{(5^2 + 12^2)} = \sqrt{169} = 13$

$|\overrightarrow{OB}| = \sqrt{(6^2 + 10^2)} = \sqrt{136} = \sqrt{4}\sqrt{34} = 2\sqrt{34}$

$|\overrightarrow{OC}| = \sqrt{((-4)^2 + 2^2)} = \sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$

$|\overrightarrow{OD}| = \sqrt{((-3)^2 + (-1)^2)} = \sqrt{10}$

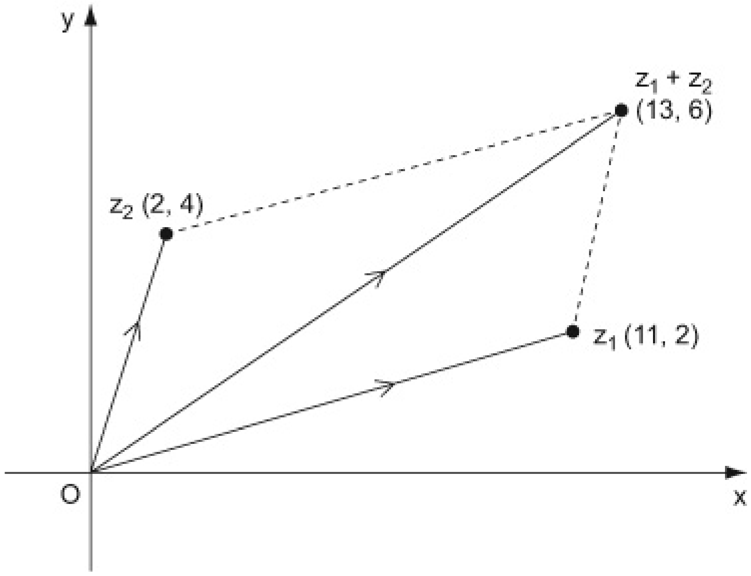
Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Complex numbers
Exercise D, Question 5

Question:

$z_1 = 11 + 2i$ and $z_2 = 2 + 4i$. Show z_1, z_2 and $z_1 + z_2$ on an Argand diagram.

Solution:



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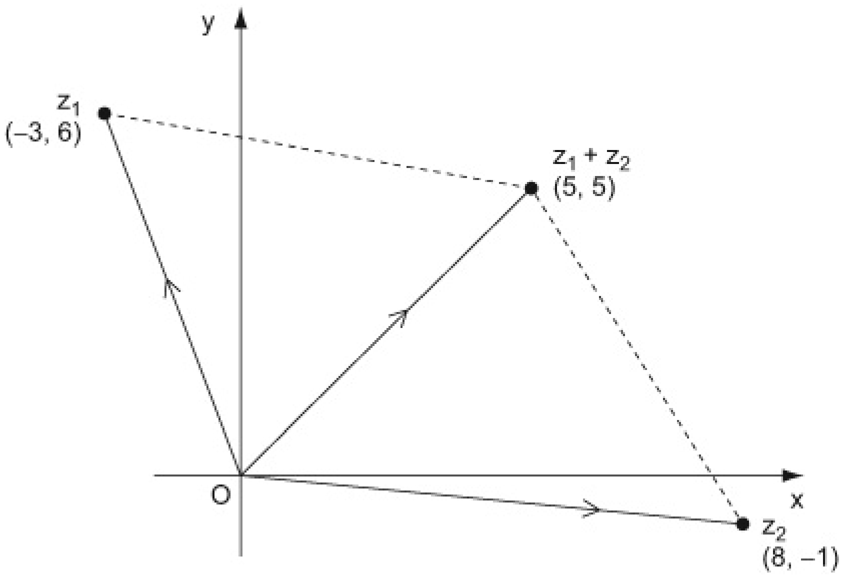
Complex numbers
Exercise D, Question 6

Question:

$z_1 = -3 + 6i$ and $z_2 = 8 - i$. Show z_1,z_2 and $z_1 + z_2$ on an Argand diagram.

Solution:



Solutionbank FP1
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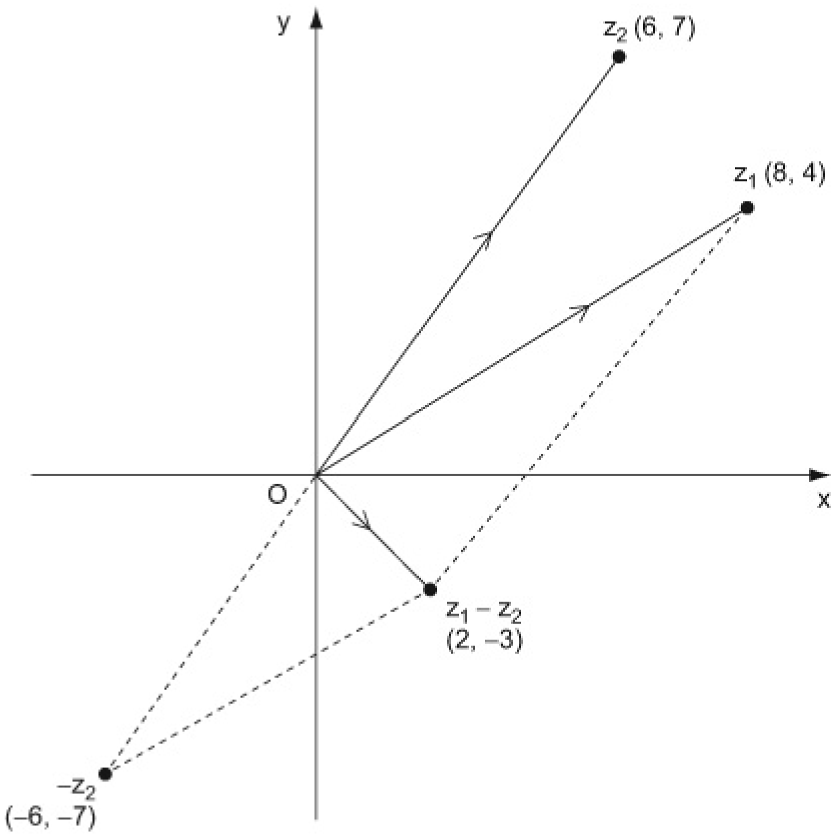
Complex numbers
Exercise D, Question 7

Question:

$z_1 = 8 + 4\text{i}$ and $z_2 = 6 + 7\text{i}$. Show z_1, z_2 and $z_1 - z_2$ on an Argand diagram.

Solution:



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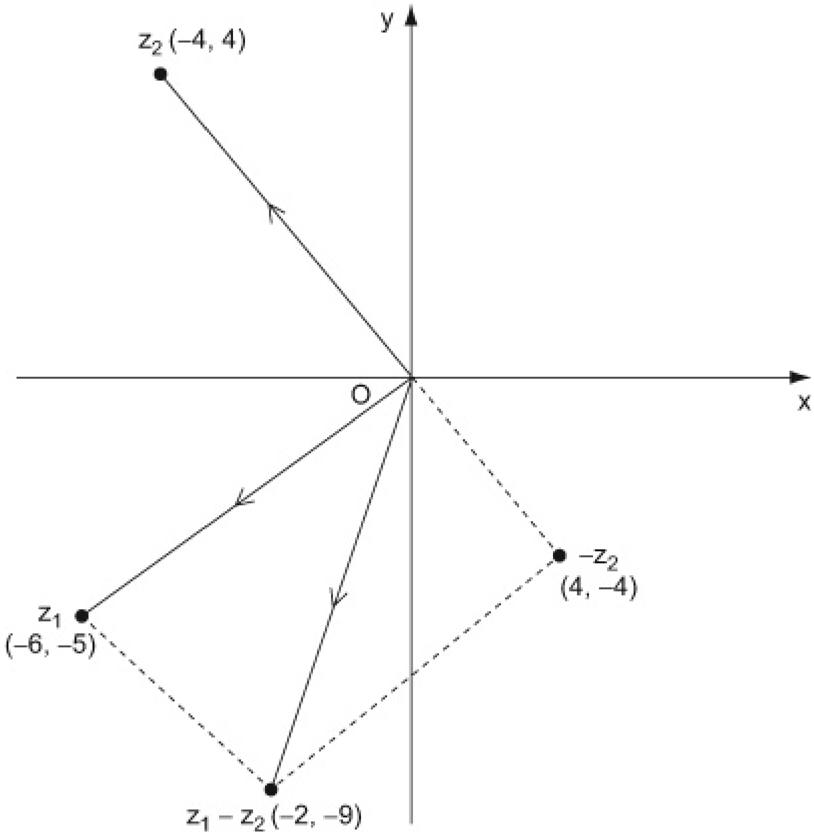
Complex numbers
Exercise D, Question 8

Question:

$z_1 = -6 - 5i$ and $z_2 = -4 + 4i$. Show z_1,z_2 and $z_1 - z_2$ on an Argand diagram.

Solution:



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Complex numbers
Exercise E, Question 1

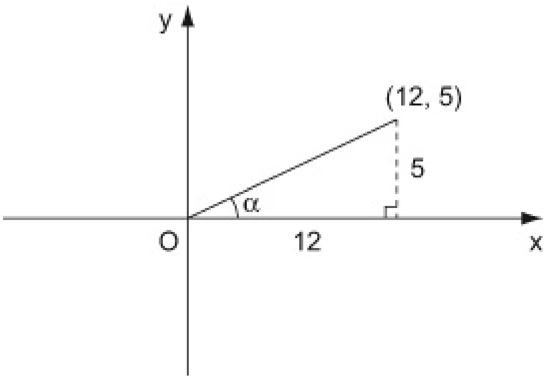
Question:

Find the modulus and argument of each of the following complex numbers, giving your answers exactly where possible, and to two decimal places otherwise.

$12 + 5\text{i}$

Solution:

$z = 12 + 5\text{i}$



$$|z| = \sqrt{(12^2 + 5^2)} = \sqrt{169} = 13$$
$$\tan\alpha = \frac{5}{12}. \quad \alpha = 0.39 \text{ rad.}$$
$$\arg z = 0.39$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Complex numbers
Exercise E, Question 2

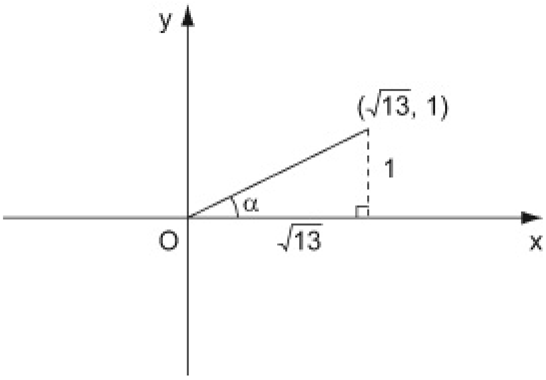
Question:

Find the modulus and argument of each of the following complex numbers, giving your answers exactly where possible, and to two decimal places otherwise.

$\sqrt{3} + \mathrm{i}$

Solution:

$z = \sqrt{3} + \mathrm{i}$



$|z| = \sqrt{\left((\sqrt{3})^2 + 1^2\right)} = \sqrt{4} = 2$

$\tan \alpha = \frac{1}{\sqrt{3}}. \quad \alpha = \frac{\pi}{6}.$

$\arg z = \frac{\pi}{6}.$

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Complex numbers
Exercise E, Question 3

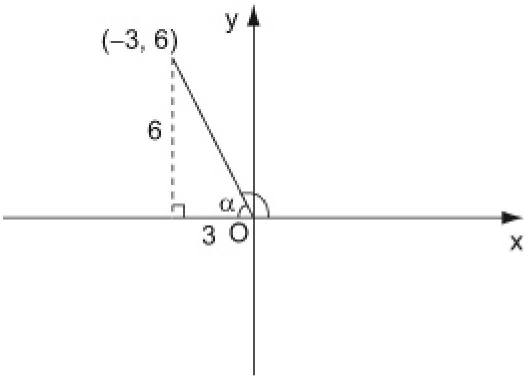
Question:

Find the modulus and argument of each of the following complex numbers, giving your answers exactly where possible, and to two decimal places otherwise.

$-3+6\text{i}$

Solution:

$z=-3+6\text{i}$



$|z|=\sqrt{\left((-3)^2+6^2\right)}=\sqrt{45}=3\sqrt{5}$

$\tan\alpha=\frac{6}{3}. \quad \alpha=1.107 \text{ rad}$

$\arg z=\pi-\alpha=2.03$

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Complex numbers
Exercise E, Question 4

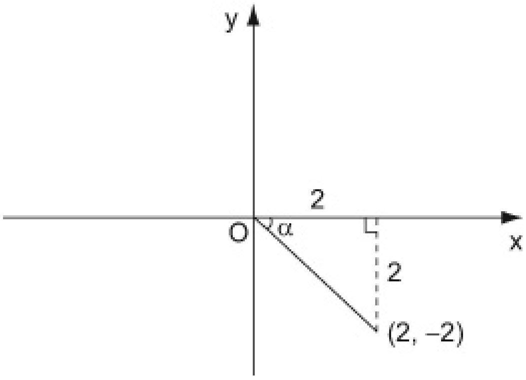
Question:

Find the modulus and argument of each of the following complex numbers, giving your answers exactly where possible, and to two decimal places otherwise.

$2 - 2i$

Solution:

$z = 2 - 2i$



$$|z| = \sqrt{\left(2^2 + (-2)^2\right)} = \sqrt{8} = 2\sqrt{2}$$

$$\tan\alpha = \frac{2}{2}. \quad \alpha = \frac{\pi}{4}.$$
$$\arg z = -\alpha = -\frac{\pi}{4}.$$

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Complex numbers
Exercise E, Question 5

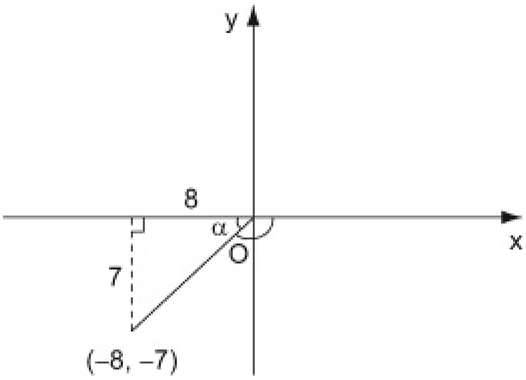
Question:

Find the modulus and argument of each of the following complex numbers, giving your answers exactly where possible, and to two decimal places otherwise.

$-8 - 7\text{i}$

Solution:

$z = -8 - 7\text{i}$



$$|z| = \sqrt{\left((-8)^2 + (-7)^2\right)} = \sqrt{113}$$

$$\tan\alpha = \frac{7}{8}. \quad \alpha = 0.7188 \text{ rad}$$
$$\arg z = -(\pi - \alpha) = -2.42$$

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Complex numbers
Exercise E, Question 6

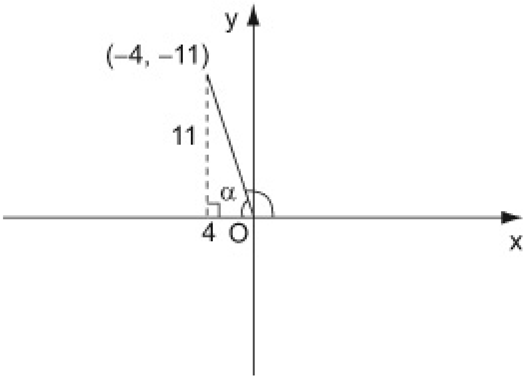
Question:

Find the modulus and argument of each of the following complex numbers, giving your answers exactly where possible, and to two decimal places otherwise.

$-4+11\text{i}$

Solution:

$z=-4+11\text{i}$



$|z|=\sqrt{\left((-4)^2+11^2\right)}=\sqrt{137}$

$\tan\alpha=\frac{11}{4}.\quad \alpha=1.222\ \text{rad}$

$\arg z=\pi-\alpha=1.92$

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Complex numbers
Exercise E, Question 7

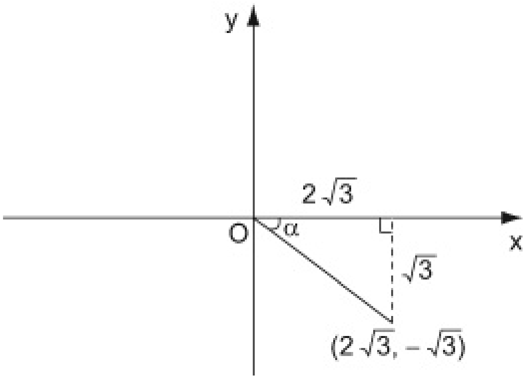
Question:

Find the modulus and argument of each of the following complex numbers, giving your answers exactly where possible, and to two decimal places otherwise.

$2\sqrt{3} - \mathrm{i}\sqrt{3}$

Solution:

$z = 2\sqrt{3} - \mathrm{i}\sqrt{3}$



$|z| = \sqrt{\left((2\sqrt{3})^2 + (-\sqrt{3})^2\right)} = \sqrt{15}$

$\tan\alpha = \dfrac{\sqrt{3}}{2\sqrt{3}} = \dfrac{1}{2}. \quad \alpha = 0.4636 \text{ rad.}$

$\arg z = -0.46$

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Complex numbers
Exercise E, Question 8

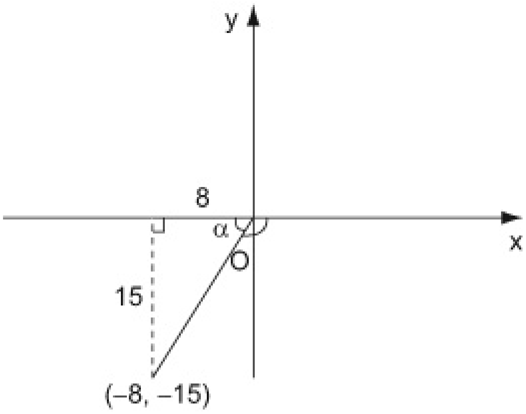
Question:

Find the modulus and argument of each of the following complex numbers, giving your answers exactly where possible, and to two decimal places otherwise.

$-8 - 15i$

Solution:

$z = -8 - 15i$



$|z| = \sqrt{\left((-8)^2 + (-15)^2\right)} = \sqrt{289} = 17$

$\tan\alpha = \frac{15}{8}. \quad \alpha = 1.0808 \text{ rad.}$

$\arg z = -(\pi - \alpha) = -2.06$

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Complex numbers
Exercise F, Question 1

Question:

Express these in the form $r(\cos\ \theta + \mathrm{i}\ \sin\ \theta)$, giving exact values of r and θ where possible, or values to two decimal places otherwise.

a $2 + 2\mathrm{i}$

b $3\mathrm{i}$

c $-3 + 4\mathrm{i}$

d $1 - \sqrt{3}\,\mathrm{i}$

e $-2 - 5\mathrm{i}$

f -20

g $7 - 24\mathrm{i}$

h $-5 + 5\mathrm{i}$

Solution:

a

$$\begin{aligned} r &= \sqrt{\left(2^2 + 2^2\right)} = \sqrt{8} = 2\sqrt{2} \\ \tan\alpha &= \frac{2}{2} = 1. \qquad \alpha = \frac{\pi}{4} \\ \theta &= \frac{\pi}{4} \\ 2 + 2\mathrm{i} &= 2\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) \end{aligned}$$

b

$$\begin{aligned} r &= \sqrt{\left(O^2 + 3^2\right)} = \sqrt{9} = 3 \\ \tan\alpha &= \infty \qquad \alpha = \frac{\pi}{2} \\ \theta &= \frac{\pi}{2} \\ 3\mathrm{i} &= 3\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right) \end{aligned}$$

c

$$\begin{aligned} r &= \sqrt{\left((-3)^2 + 4^2\right)} = \sqrt{2}5 = 5 \\ \tan\alpha &= \frac{4}{3}. \qquad \alpha = 0.927\ \text{rad.} \\ \theta &= \pi - \alpha = 2.21 \\ -3 + 4\mathrm{i} &= 5(\cos 2.21 + \mathrm{i}\sin 2.21) \end{aligned}$$

d

$$\begin{aligned} r &= \sqrt{\left(1^2 + (-\sqrt{3})^2\right)} = \sqrt{4} = 2 \\ \tan\alpha &= \frac{\sqrt{3}}{1}. \qquad \alpha = \frac{\pi}{3} \\ \theta &= -\frac{\pi}{3} \\ 1 - \sqrt{3}\,\mathrm{i} &= 2\left(\cos\left(\frac{-\pi}{3}\right) + \mathrm{i}\sin\left(\frac{-\pi}{3}\right)\right). \end{aligned}$$

e

$$\begin{aligned} r &= \sqrt{\left((-2)^2 + (-5)^2\right)} = \sqrt{29} \\ \tan\alpha &= \frac{5}{2}. \qquad \alpha = 1.190 \text{ rad} \\ \theta &= -(\pi - \alpha) = -1.95 \\ -2 - 5\mathrm{i} &= \sqrt{29}\,(\cos(-1.95) + \mathrm{i}\sin(-1.95)). \end{aligned}$$

f

$$\begin{aligned} r &= \sqrt{\left((-20)^2 + O^2\right)} = \sqrt{400} = 20 \\ \tan\alpha &= O \\ \theta &= \pi \\ -20 &= 20(\cos\pi + \mathrm{i}\sin\pi) \end{aligned}$$

g

$$\begin{aligned} r &= \sqrt{\left(7^2 + (-24)^2\right)} = \sqrt{625} = 25 \\ \tan\alpha &= \frac{24}{7}. \qquad \alpha = 1.287 \text{ rad} \\ \theta &= -1.29 \\ 7 - 24\mathrm{i} &= 25(\cos(-1.29) + \mathrm{i}\sin(-1.29)) \end{aligned}$$

h

$$\begin{aligned} r &= \sqrt{\left((-5)^2 + 5^2\right)} = \sqrt{50} = 5\sqrt{2} \\ \tan\alpha &= \frac{5}{5} = 1. \qquad \alpha = \frac{\pi}{4}. \\ \theta &= \pi - \alpha = \frac{3\pi}{4} \\ -5 + 5\mathrm{i} &= 5\sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right). \end{aligned}$$

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Complex numbers
Exercise F, Question 2

Question:

Express these in the form $r(\cos\ \theta + \mathrm{i}\ \sin\ \theta)$, giving exact values of r and θ where possible, or values to two decimal places otherwise.

a $\frac{3}{1+\mathrm{i}\sqrt{3}}$

b $\frac{1}{2-\mathrm{i}}$

c $\frac{1+\mathrm{i}}{1-\mathrm{i}}$

Solution:

a

$$
\begin{aligned}
\frac{3}{1+\mathrm{i}\sqrt{3}} &= \frac{3(1-\mathrm{i}\sqrt{3})}{(1+\mathrm{i}\sqrt{3})(1-\mathrm{i}\sqrt{3})} \\
&= \frac{3-3\mathrm{i}\sqrt{3}}{1(1-\mathrm{i}\sqrt{3})+\mathrm{i}\sqrt{3}(1-\mathrm{i}\sqrt{3})} \\
&= \frac{3-3\mathrm{i}\sqrt{3}}{1-\mathrm{i}\sqrt{3}+\mathrm{i}\sqrt{3}-3\mathrm{i}^2} = \frac{3-3\mathrm{i}\sqrt{3}}{4} \\
&= \frac{3}{4}-\frac{3\sqrt{3}}{4}\mathrm{i} \\
r &= \sqrt{\left(\left(\frac{3}{4}\right)^2+\left(-\frac{3\sqrt{3}}{4}\right)^2\right)} = \sqrt{\left(\frac{9}{16}+\frac{27}{16}\right)} \\
&= \sqrt{\left(\frac{36}{16}\right)} = \frac{3}{2} \\
\tan\alpha &= \frac{3\sqrt{3}}{4} \div \frac{3}{4} = \sqrt{3}. \qquad \alpha = \frac{\pi}{3} \\
\theta &= -\frac{\pi}{3} \\
\frac{3}{1+\mathrm{i}\sqrt{3}} &= \frac{3}{2}\left(\cos\left(\frac{-\pi}{3}\right)+\mathrm{i}\sin\left(\frac{-\pi}{3}\right)\right)
\end{aligned}
$$

b

$$
\begin{aligned}
\frac{1}{2-\mathrm{i}} &= \frac{2+\mathrm{i}}{(2-\mathrm{i})(2+\mathrm{i})} \\
&= \frac{2+\mathrm{i}}{2(2+\mathrm{i})-\mathrm{i}(2+\mathrm{i})} = \frac{2+\mathrm{i}}{4+2\mathrm{i}-2\mathrm{i}-\mathrm{i}^2} \\
&= \frac{2+\mathrm{i}}{5} = \frac{2}{5}+\frac{1}{5}\mathrm{i} \\
r &= \sqrt{\left(\left(\frac{2}{5}\right)^2+\left(\frac{1}{5}\right)^2\right)} = \sqrt{\left(\frac{4}{25}+\frac{1}{25}\right)} \\
&= \sqrt{\frac{5}{25}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} \\
\tan\alpha &= \frac{1}{5} \div \frac{2}{5} = \frac{1}{2}. \qquad \alpha = 0.4636\ \text{rad.} \\
\theta &= 0.46 \\
\frac{1}{2-\mathrm{i}} &= \frac{\sqrt{5}}{5}(\cos\ 0.46+\mathrm{i}\sin\ 0.46)
\end{aligned}
$$

c

$$\frac{1+\mathrm{i}}{1-\mathrm{i}} = \frac{(1+\mathrm{i})(1+\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})}$$

$$= \frac{1(1+\mathrm{i})+\mathrm{i}(1+\mathrm{i})}{1(1+\mathrm{i})-\mathrm{i}(1+\mathrm{i})} = \frac{1+\mathrm{i}+\mathrm{i}+\mathrm{i}^2}{1+\mathrm{i}-\mathrm{i}-\mathrm{i}^2}$$

$$= \frac{2\mathrm{i}}{2} = \mathrm{i}$$

$$r = \sqrt{\left(0^2+1^2\right)} = 1$$

$$\tan\alpha = \infty. \qquad \alpha = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{2}$$

$$\frac{1+\mathrm{i}}{1-\mathrm{i}} = 1\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$$

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Complex numbers
Exercise F, Question 3

Question:

Write in the form $a + \mathrm{i}b$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

a $3\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\ \sin\frac{\pi}{4}\right)$

b $6\left(\cos\frac{3\pi}{4} + \mathrm{i}\ \sin\frac{3\pi}{4}\right)$

c $\sqrt{3}\left(\cos\frac{\pi}{3} + \mathrm{i}\ \sin\frac{\pi}{3}\right)$

d $7\left(\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\ \sin\left(-\frac{\pi}{2}\right)\right)$

e $4\left(\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\ \sin\left(-\frac{5\pi}{6}\right)\right)$

Solution:

a $3\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right) = 3 + 3\mathrm{i}$

b

$$\begin{aligned} 6\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right) &= \frac{-6}{\sqrt{2}} + \frac{6}{\sqrt{2}}\mathrm{i} \\ &= -3\sqrt{2} + 3\sqrt{2}\,\mathrm{i} \end{aligned}$$

c $\sqrt{3}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) = \frac{\sqrt{3}}{2} + \frac{3}{2}\mathrm{i}$

d $7(0 + (-1)\mathrm{i}) = -7\mathrm{i}$

e $4\left(\frac{-\sqrt{3}}{2} + \left(\frac{-1}{2}\right)\mathrm{i}\right) = -2\sqrt{3} - 2\mathrm{i}$

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Complex numbers
Exercise F, Question 4

Question:

In each case, find $|z_1|, |z_2|$ and z_1z_2, and verify that $|z_1z_2| = |z_1||z_2|$.

a $z_1 = 3 + 4\text{i}$ $\quad z_2 = 4 - 3\text{i}$

b $z_1 = -1 + 2\text{i}$ $\quad z_2 = 4 + 2\text{i}$

c $z_1 = 5 + 12\text{i}$ $\quad z_2 = 7 + 24\text{i}$

d $z_1 = \sqrt{3} + \text{i}\sqrt{2}$ $\quad z_2 = -\sqrt{2} + \text{i}\sqrt{3}$

Solution:

a

$$\begin{aligned}
|z_1| &= \sqrt{\left(3^2 + 4^2\right)} = \sqrt{25} = 5 \\
|z_2| &= \sqrt{\left(4^2 + (-3)^2\right)} = \sqrt{25} = 5 \\
z_1z_2 &= (3 + 4\text{i})(4 - 3\text{i}) \\
&= 3(4 - 3\text{i}) + 4\text{i}(4 - 3\text{i}) \\
&= 12 - 9\text{i} + 16\text{i} - 12\text{i}^2 \\
&= 24 + 7\text{i} \\
|z_1z_2| &= \sqrt{\left(24^2 + 7^2\right)} = \sqrt{625} = 25 \\
|z_1||z_2| &= 5 \times 5 = 25 = |z_1z_2|
\end{aligned}$$

b

$$\begin{aligned}
|z_1| &= \sqrt{\left((-1)^2 + 2^2\right)} = \sqrt{5} \\
|z_2| &= \sqrt{\left(4^2 + 2^2\right)} = \sqrt{20} = 2\sqrt{5} \\
z_1z_2 &= (-1 + 2\text{i})(4 + 2\text{i}) \\
&= -1(4 + 2\text{i}) + 2\text{i}(4 + 2\text{i}) \\
&= -4 - 2\text{i} + 8\text{i} + 4\text{i}^2 \\
&= -8 + 6\text{i} \\
|z_1z_2| &= \sqrt{\left((-8)^2 + 6^2\right)} = \sqrt{100} = 10 \\
|z_1||z_2| &= \sqrt{5} \times 2\sqrt{5} = 10 = |z_1z_2|
\end{aligned}$$

c

$$|z_1| = \sqrt{\left(5^2 + 12^2\right)} = \sqrt{169} = 13$$

$$|z_2| = \sqrt{\left(7^2 + 24^2\right)} = \sqrt{625} = 25$$

$$\begin{aligned} z_1z_2 &= (5+12\text{i})(7+24\text{i}) \\ &= 5(7+24\text{i}) + 12\text{i}(7+24\text{i}) \\ &= 35 + 120\text{i} + 84\text{i} + 288\text{i}^2 \\ &= -253 + 204\text{i} \end{aligned}$$

$$|z_1z_2| = \sqrt{\left((-253)^2 + 204^2\right)} = \sqrt{105625} = 325$$

$$|z_1||z_2| = 13 \times 25 = 325 = |z_1z_2|$$

d

$$|z_1| = \sqrt{\left((\sqrt{3})^2 + (\sqrt{2})^2\right)} = \sqrt{5}$$

$$|z_2| = \sqrt{\left((-\sqrt{2})^2 + (\sqrt{3})^2\right)} = \sqrt{5}$$

$$\begin{aligned} z_1z_2 &= (\sqrt{3} + \text{i}\sqrt{2})(-\sqrt{2} + \text{i}\sqrt{3}) \\ &= \sqrt{3}(-\sqrt{2} + \text{i}\sqrt{3}) + \text{i}\sqrt{2}(-\sqrt{2} + \text{i}\sqrt{3}) \\ &= -\sqrt{6} + 3\text{i} - 2\text{i} + \text{i}^2\sqrt{6} \\ &= -2\sqrt{6} + \text{i} \end{aligned}$$

$$|z_1z_2| = \sqrt{\left((-2\sqrt{6})^2 + 1^2\right)} = \sqrt{(24+1)} = 5$$

$$|z_1||z_2| = \sqrt{5} \times \sqrt{5} = 5 = |z_1z_2|.$$

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Complex numbers
Exercise G, Question 1

Question:

$a + 2b + 2a\text{i} = 4 + 6\text{i}$, where a and b are real.

Find the value of a and the value of b.

Solution:

Real parts: $a + 2b = 4$

Imaginary parts: $2a = 6$

$a = 3$

$$\begin{aligned} 3 + 2b &= 4 \\ 2b &= 1 \\ b &= \frac{1}{2} \end{aligned}$$

$a = 3$ and $b = \frac{1}{2}$

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Complex numbers
Exercise G, Question 2

Question:

$(a-b)+(a+b)\mathrm{i}=9+5\mathrm{i}$, where a and b are real.

Find the value of a and the value of b.

Solution:

Real parts : $a-b=9$
Imaginary parts : $a+b=5$
Adding : $2a=14$
$a=7$

$7-b=9$
$b=-2$

$a=7$ and $b=-2$.

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Complex numbers
Exercise G, Question 3

Question:

$(a+b)(2+\mathrm{i}) = b+1+(10+2a)\mathrm{i}$, where a and b are real.

Find the value of a and the value of b.

Solution:

Real parts : $2(a+b) = b+1$

$2a+2b = b+1$

$2a+b = 1$ (i)

Imaginary parts : $a+b = 10+2a$

$-a+b = 10$ (ii)

(i) − (ii) : $3a = -9$

$a = -3$

Substitute into (i) : $-6+b = 1$

$b = 7$

$a=-3$ and $b=7$

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Complex numbers
Exercise G, Question 4

Question:

$(a+\mathrm{i})^3 = 18 + 26\mathrm{i}$, where a is real.

Find the value of a.

Solution:

$$\begin{aligned}(a+\mathrm{i})^3 &= a^3 + 3a^2\mathrm{i} + 3a\mathrm{i}^2 + \mathrm{i}^3\\ &= (a^3 - 3a) + \mathrm{i}(3a^2 - 1)\end{aligned}$$

Imaginary part : $3a^2 - 1 = 26$
$$\begin{aligned}3a^2 &= 27\\ a^2 &= 9\\ a &= 3 \text{ or } -3\end{aligned}$$

Real part : $a = 3$ gives $27 - 9 = 18$. Correct.
$a = -3$ gives $-27 + 9 = -18$. Wrong.

So $a = 3$.

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Complex numbers
Exercise G, Question 5

Question:

$ab\text{i} = 3a - b + 12\text{i}$, where a and b are real.

Find the value of a and the value of b.

Solution:

Real parts: $O = 3a - b$ (i)

Imaginary parts : $ab = 12$ (ii)

From (ii), $b = \frac{12}{a}$

Substitute into (i) : $O = 3a - \frac{12}{a}$

$$3a^2 - 12 = 0$$
$$a^2 = 4$$
$$a = 2 \text{ or } -2$$

If $a = 2$, $b = \frac{12}{2} = 6$

If $a = -2$, $b = \frac{12}{-2} = -6$

Either $a = 2$ and $b = 6$
or $a = -2$ and $b = -6$.

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Complex numbers
Exercise G, Question 6

Question:

Find the real numbers x and y, given that

$$\frac{1}{x+\mathrm{i}y}=3-2\mathrm{i}$$

Solution:

$$(3-2\mathrm{i})(x+\mathrm{i}y)=1$$

$$3(x+\mathrm{i}y)-2\mathrm{i}(x+\mathrm{i}y)=1$$
$$3x+3y\mathrm{i}-2x\mathrm{i}-2\mathrm{i}^2y=1$$
$$(3x+2y)+\mathrm{i}(3y-2x)=1$$

Real parts: $3x+2y=1$ (i)

Imaginary parts : $3y-2x=0$ (ii)

$2\times$(i) $+\,3\times$(ii):

$$\begin{aligned}6x+4y+9y-6x&=2\\13y&=2\\y&=\frac{2}{13}\end{aligned}$$

Substitute into (i) : $3x+\frac{4}{13}=1$

$$\begin{aligned}3x&=\frac{9}{13}.\\x&=\frac{3}{13}\end{aligned}$$

$x=\frac{3}{13}$ and $y=\frac{2}{13}$

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Complex numbers
Exercise G, Question 7

Question:

Find the real numbers x and y, given that

$(x+\mathrm{i}y)(1+\mathrm{i})=2+\mathrm{i}$

Solution:

$$\begin{aligned}(x+\mathrm{i}y)(1+\mathrm{i}) &= x(1+\mathrm{i})+\mathrm{i}y(1+\mathrm{i})\\ &= x+x\mathrm{i}+\mathrm{i}y+\mathrm{i}^2y\\ &= (x-y)+\mathrm{i}(x+y)\end{aligned}$$

Real parts : $x-y=2$
Imaginary parts : $x+y=1$

Adding : $2x=3$

$x=\frac{3}{2}$

$\frac{3}{2}+y=1$, $y=-\frac{1}{2}$

$x=\frac{3}{2}$ and $y=-\frac{1}{2}$

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Complex numbers
Exercise G, Question 8

Question:

Solve for real x and y

$(x + iy)(5 - 2i) = -3 + 7i$

Hence find the modulus and argument of $x + iy$.

Solution:

$$\begin{aligned}(x+iy)(5-2i) &= x(5-2i) + iy(5-2i)\\ &= 5x - 2xi + 5yi - 2yi^2\\ &= (5x+2y) + i(-2x+5y)\end{aligned}$$

Real parts: $5x + 2y = -3$ (i)

Imaginary parts : $-2x + 5y = 7$ (ii)

$$\begin{aligned}\text{(i)}\times 2: \quad 10x + 4y &= -6\\ \text{(ii)}\times 5: \quad -10x + 25y &= 35\\ \text{Adding}: \quad 29y &= 29\\ y &= 1\end{aligned}$$

$$\begin{aligned}\text{Substitute into (i)}: 5x + 2 &= -3\\ 5x &= -5\\ x &= -1\end{aligned}$$

$x = -1$ and $y = 1$

$|-1 + i| = \sqrt{((-1)^2 + 1^2)} = \sqrt{2}$

$$\begin{aligned}\arg(-1+i) &= \pi - \arctan 1\\ &= \pi - \frac{\pi}{4} = \frac{3\pi}{4}\end{aligned}$$

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Complex numbers
Exercise G, Question 9

Question:

Find the square roots of $7 + 24\text{i}$.

Solution:

$$\begin{aligned}(a+\text{i}b)^2 &= 7+24\text{i}\\ a(a+\text{i}b)+\text{i}b(a+\text{i}b) &= 7+24\text{i}\\ a^2+ab\text{i}+ab\text{i}+b^2\text{i}^2 &= 7+24\text{i}\\ (a^2-b^2)+2ab\text{i} &= 7+24\text{i}\end{aligned}$$

Real parts: $a^2 - b^2 = 7$ (i)

Imaginary parts: $2ab = 24$ (ii)

From (ii), $b = \frac{24}{2a} = \frac{12}{a}$

Substituting into (i) : $a^2 - \frac{144}{a^2} = 7$

$$\begin{aligned}a^4 - 144 &= 7a^2\\ a^4 - 7a^2 - 144 &= 0\\ (a^2-16)(a^2+9) &= 0\\ a^2 = 16 \text{ or } a^2 &= -9\end{aligned}$$

Since a is real, $a = 4$ or $a = -4$

When $a = 4, b = \frac{12}{a} = \frac{12}{4} = 3$

When $a = -4, b = \frac{12}{-4} = -3$

Square roots are $4 + 3\text{i}$ and $-(4+3\text{i})$, i .e . $\pm(4+3\text{i})$

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Complex numbers
Exercise G, Question 10

Question:

Find the square roots of $11 + 60\text{i}$.

Solution:

$$\begin{aligned}(a + \text{i}b)^2 &= 11 + 60\text{i}\\ a(a + \text{i}b) + \text{i}b(a + \text{i}b) &= 11 + 60\text{i}\\ a^2 + ab\text{i} + ab\text{i} + b^2\text{i}^2 &= 11 + 60\text{i}\\ (a^2 - b^2) + 2ab\text{i} &= 11 + 60\text{i}\end{aligned}$$

Real parts: $a^2 - b^2 = 11$ (i)

Imaginary parts: $2ab = 60$ (ii)

From (ii): $b = \frac{60}{2a} = \frac{30}{a}$

Substituting into (i): $a^2 - \frac{900}{a^2} = 11$

$$\begin{aligned}a^4 - 900 &= 11a^2\\ a^4 - 11a^2 - 900 &= 0\\ (a^2 - 36)(a^2 + 25) &= 0\\ a^2 = 36 \text{ or } a^2 &= -25\end{aligned}$$

Since a is real, $a = 6$ or $a = -6$.

When $a = 6, b = \frac{30}{a} = \frac{30}{6} = 5$

When $a = -6, b = \frac{30}{-6} = -5$.

Square roots are $6 + 5\text{i}$ and $-(6 + 5\text{i})$,

i. e. $\pm(6 + 5\text{i})$

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Complex numbers
Exercise G, Question 11

Question:

Find the square roots of $5-12\text{i}$.

Solution:

$$\begin{aligned}(a+\text{i}b)^2 &= 5-12\text{i}\\ a(a+\text{i}b)+\text{i}b(a+\text{i}b) &= 5-12\text{i}\\ a^2+ab\text{i}+ab\text{i}+b^2\text{i}^2 &= 5-12\text{i}\\ (a^2-b^2)+2ab\text{i} &= 5-12\text{i}\end{aligned}$$

Real parts: $a^2-b^2=5$ (i)

Imaginary parts: $2ab=-12$ (ii)

From (ii): $b=\frac{-12}{2a}=\frac{-6}{a}$

Substituting into (i): $a^2-\frac{36}{a^2}=5$

$$\begin{aligned}a^4-36&=5a^2\\ a^4-5a^2-36&=0\\ (a^2-9)(a^2+4)&=0\\ a^2=9 \text{ or } a^2&=-4.\end{aligned}$$

Since a is real, $a=3$ or $a=-3$

When $a=3, b=\frac{-6}{a}=\frac{-6}{3}=-2$

When $a=-3, b=\frac{-6}{-3}=2$

Square roots are $3-2\text{i}$ and $-(3-2\text{i})$,

i. e. $\pm(3-2\text{i})$

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Complex numbers
Exercise G, Question 12

Question:

Find the square roots of 2i.

Solution:

$$(a+\mathrm{i}b)^2 = 2\mathrm{i}$$
$$a(a+\mathrm{i}b)+\mathrm{i}b(a+\mathrm{i}b) = 2\mathrm{i}$$
$$a^2+ab\mathrm{i}+ab\mathrm{i}+b^2\mathrm{i}^2 = 2\mathrm{i}$$
$$(a^2-b^2)+2ab\mathrm{i} = 2\mathrm{i}$$

Real parts: $a^2-b^2=0$ (i)

Imaginary parts: $2ab=2$ (ii)

From (ii): $b=\frac{2}{2a}=\frac{1}{a}$

Substituting into (i): $a^2-\frac{1}{a^2}=0$

$$a^4-1=0$$
$$a^4=1$$

Real solutions are $a=1$ or $a=-1$.

When $a=1, b=\frac{1}{a}=\frac{1}{1}=1$

When $a=-1, b=\frac{1}{-1}=-1$.

Square roots are $1+\mathrm{i}$ and $-(1+\mathrm{i})$,

i. e. $\pm(1+\mathrm{i})$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Complex numbers
Exercise H, Question 1

Question:

Given that $1 + 2\mathrm{i}$ is one of the roots of a quadratic equation, find the equation.

Solution:

The other root is $1 - 2\mathrm{i}$.

If the roots are α and β, the equation is

$$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

$$\begin{aligned}\alpha+\beta &= (1+2\mathrm{i}) + (1-2\mathrm{i}) = 2\\ \alpha\beta &= (1+2\mathrm{i})(1-2\mathrm{i})\\ &= 1(1-2\mathrm{i}) + 2\mathrm{i}(1-2\mathrm{i})\\ &= 1 - 2\mathrm{i} + 2\mathrm{i} - 4\mathrm{i}^2 = 5\end{aligned}$$

Equation is $x^2 - 2x + 5 = 0$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Complex numbers
Exercise H, Question 2

Question:

Given the $3-5\mathrm{i}$ is one of the roots of a quadratic equation, find the equation.

Solution:

The other root is $3+5\mathrm{i}$.

If the roots are α and β, the equation is

$$(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta=0.$$

$$\begin{aligned}\alpha+\beta &= (3-5\mathrm{i})+(3+5\mathrm{i})=6\\ \alpha\beta &= (3-5\mathrm{i})(3+5\mathrm{i})\\ &= 3(3+5\mathrm{i})-5\mathrm{i}(3+5\mathrm{i})\\ &= 9+15\mathrm{i}-15\mathrm{i}-25\mathrm{i}^2=34\end{aligned}$$

Equation is $x^2-6x+34=0$

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Complex numbers
Exercise H, Question 3

Question:

Given that $a + 4\mathrm{i}$, where a is real, is one of the roots of a quadratic equation, find the equation.

Solution:

The other root is $a - 4\mathrm{i}$.

If the roots are α and β, the equation is

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta = 0.$$

$$\begin{aligned}
\alpha + \beta &= (a + 4\mathrm{i}) + (a - 4\mathrm{i}) = 2a \\
\alpha\beta &= (a + 4\mathrm{i})(a - 4\mathrm{i}) \\
&= a(a - 4\mathrm{i}) + 4\mathrm{i}(a - 4\mathrm{i}) \\
&= a^2 - 4a\mathrm{i} + 4a\mathrm{i} - 16\mathrm{i}^2 = a^2 + 16
\end{aligned}$$

Equation is $x^2 - 2ax + a^2 + 16 = 0$

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Complex numbers
Exercise H, Question 4

Question:

Show that $x = -1$ is a root of the equation $x^3 + 9x^2 + 33x + 25 = 0$.

Hence solve the equation completely.

Solution:

When $x = -1$,

$x^3 + 9x^2 + 33x + 25 = -1 + 9 - 33 + 25 = 0$

So $x = -1$ is a root.

So $(x + 1)$ is a factor

$x^3 + 9x^2 + 33x + 25 = (x + 1)(x^2 + 8x + 25) = 0$

$a = 1,\ b = 8,\ c = 25.$

$$x = \frac{-8 \pm \sqrt{(64 - 100)}}{2} = \frac{-8 \pm 6i}{2} = -4 \pm 3i$$

Roots are -1, $-4 + 3i$ and $-4 - 3i$

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Complex numbers
Exercise H, Question 5

Question:

Show that $x = 3$ is a root of the equation $2x^3 - 4x^2 - 5x - 3 = 0$.

Hence solve the equation completely.

Solution:

When $x = 3$,

$2x^3 - 4x^2 - 5x - 3 = 54 - 36 - 15 - 3 = 0$.

So $x = 3$ is a root.

So $(x - 3)$ is a factor.

$2x^3 - 4x^2 - 5x - 3 = (x - 3)(2x^2 + 2x + 1) = 0$

$a = 2,\ b = 2,\ c = 1.$

$x = \dfrac{-2 \pm \sqrt{(4-8)}}{4} = \dfrac{-2 \pm 2\mathrm{i}}{4} = \dfrac{-1}{2} \pm \dfrac{1}{2}\mathrm{i}$

Roots are 3, $\dfrac{-1}{2} + \dfrac{1}{2}\mathrm{i}$ and $\dfrac{-1}{2} - \dfrac{1}{2}\mathrm{i}$

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Complex numbers
Exercise H, Question 6

Question:

Show that $x = -\frac{1}{2}$ is a root of the equation $2x^3 + 3x^2 + 3x + 1 = 0$.

Hence solve the equation completely.

Solution:

When $x = \frac{-1}{2}$,

$$2x^3 + 3x^2 + 3x + 1 = 2\left(\frac{-1}{8}\right) + 3\left(\frac{1}{4}\right) + 3\left(\frac{-1}{2}\right) + 1$$
$$= \frac{-1}{4} + \frac{3}{4} - \frac{3}{2} + 1 = 0$$

So $x = -\frac{1}{2}$ is a root.

So $(2x + 1)$ is a factor.

$$2x^3 + 3x^2 + 3x + 1 = (2x + 1)(x^2 + x + 1) = 0$$

$a = 1,\ b = 1,\ c = 1$

$$x = \frac{-1 \pm \sqrt{(1-4)}}{2} = \frac{-1 \pm \mathrm{i}\sqrt{3}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$$

Roots are $\frac{-1}{2}$, $\frac{-1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$ and $\frac{-1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$.

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Complex numbers
Exercise H, Question 7

Question:

Given that $-4+\mathrm{i}$ is one of the roots of the equation $x^3+4x^2-15x-68=0$, solve the equation completely.

Solution:

Another root is $-4-\mathrm{i}$

The equation with roots α and β is

$(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta=0.$

$$\begin{aligned}\alpha+\beta &= (-4+\mathrm{i})+(-4-\mathrm{i})=-8\\ \alpha\beta &= (-4+\mathrm{i})(-4-\mathrm{i})\\ &= -4(-4-\mathrm{i})+\mathrm{i}(-4-\mathrm{i})\\ &= 16+4\mathrm{i}-4\mathrm{i}-\mathrm{i}^2=17\end{aligned}$$

Quadratiz equation is $x^2+8x+17=0$.

So $(x^2+8x+17)$ is a factor of $(x^3+4x^2-15x-68)$.

$(x^3+4x^2-15x-68)=(x^2+8x+17)(x-4)$

Roots are 4, $-4+\mathrm{i}$ and $-4-\mathrm{i}$.

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Complex numbers
Exercise H, Question 8

Question:

Given that $x^4 - 12x^3 + 31x^2 + 108x - 360 = (x^2 - 9)(x^2 + bx + c)$, find the values of b and c, and hence find all the solutions of the equation $x^4 - 12x^3 + 31x^2 + 108x - 360 = 0$.

Solution:

$x^4 - 12x^3 + 31x^2 + 108x - 360 = (x^2 - 9)(x^2 + bx + c)$

x^3 terms : $-12 = b$
$b = -12$

Constant term : $-360 = -9c$
$c = 40$

$(x^2 - 9)(x^2 - 12x + 40) = 0$

$x^2 - 9 = 0$: $x^2 = 9$
$x = 3$ or $x = -3$

$x^2 - 12x + 40 = 0$

$a = 1,\ b = -12,\ c = 40$

$x = \dfrac{12 \pm \sqrt{(144 - 160)}}{2} = \dfrac{12 \pm 4\mathrm{i}}{2} = 6 \pm 2\mathrm{i}$

Roots are 3, -3, $6 + 2\mathrm{i}$ and $6 - 2\mathrm{i}$

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Complex numbers
Exercise H, Question 9

Question:

Given that $2+3\mathrm{i}$ is one of the roots of the equation $x^4+2x^3-x^2+38x+130=0$, solve the equation completely.

Solution:

Another root is $2-3\mathrm{i}$

The equation with roots α and β is

$$(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta=0$$

$$\begin{aligned}\alpha+\beta &= (2+3\mathrm{i})+(2-3\mathrm{i})=4\\ \alpha\beta &= (2+3\mathrm{i})(2-3\mathrm{i})\\ &= 2(2-3\mathrm{i})+3\mathrm{i}(2-3\mathrm{i})\\ &= 4-6\mathrm{i}+6\mathrm{i}-9\mathrm{i}^2=13\end{aligned}$$

Quadratic equation is $x^2-4x+13=0$.

So $(x^2-4x+13)$ is a factor of $(x^4+2x^3-x^2+38x+130)$.

$$(x^4+2x^3-x^2+38x+130)=(x^2-4x+13)(x^2+6x+10)$$

$$x^2+6x+10=0$$

$a=1,\ b=6,\ c=10$

$$x=\frac{-6\pm\sqrt{(36-40)}}{2}=\frac{-6\pm 2\mathrm{i}}{2}=-3\pm\mathrm{i}$$

Roots are $2+3\mathrm{i},\ 2-3\mathrm{i},\ -3+\mathrm{i}$ and $-3-\mathrm{i}$.

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Complex numbers
Exercise H, Question 10

Question:

Find the four roots of the equation $x^4 - 16 = 0$.

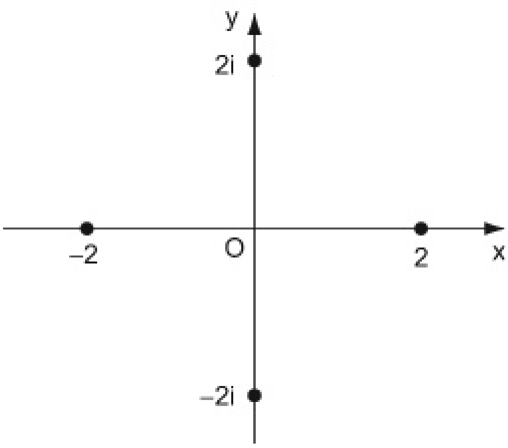
Show these roots on an Argand diagram.

Solution:

$x^4 - 16 = 0$
$(x^2 - 4)(x^2 + 4) = 0$

$x^2 = 4$ or $x^2 = -4$

$x = 2$, -2, $2i$ or $-2i$



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Complex numbers
Exercise H, Question 11

Question:

Three of the roots of the equation $ax^5 + bx^4 + cx^3 + dx^2 + ex + \mathrm{f} = 0$ are -2, $2\mathrm{i}$ and $1 + \mathrm{i}$. Find the values of *a, b, c, d, e* and *f*.

Solution:

The other two roots are $-2\mathrm{i}$ and $1 - \mathrm{i}$

The equation with roots α and β is

$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta = 0.$

Using $2i$ and $-2\mathrm{i}$,

$$\begin{aligned} \alpha + \beta &= 2\mathrm{i} - 2\mathrm{i} = 0 \\ \alpha\beta &= (2\mathrm{i})(-2\mathrm{i}) = -4\mathrm{i}^2 = 4 \end{aligned}$$

Quadratic equation is $x^2 + 4 = 0$

Using $1 + \mathrm{i}$ and $1 - \mathrm{i}$,

$$\begin{aligned} \alpha + \beta &= (1 + \mathrm{i}) + (1 - \mathrm{i}) = 2 \\ \alpha\beta &= (1 + \mathrm{i})(1 - \mathrm{i}) \\ &= 1(1 - \mathrm{i}) + \mathrm{i}(1 - \mathrm{i}) \\ &= 1 - \mathrm{i} + \mathrm{i} - \mathrm{i}^2 = 2. \end{aligned}$$

Quadratic equation is $x^2 - 2x + 2 = 0$

The required equation is

$$(x + 2)(x^2 + 4)(x^2 - 2x + 2) = 0$$
$$(x^3 + 2x^2 + 4x + 8)(x^2 - 2x + 2) = 0$$

$$x^3(x^2 - 2x + 2) + 2x^2(x^2 - 2x + 2) + 4x(x^2 - 2x + 2) + 8(x^2 - 2x + 2) = 0$$
$$x^5 - 2x^4 + 2x^3 + 2x^4 - 4x^3 + 4x^2 + 4x^3 - 8x^2 + 8x + 8x^2 - 16x + 16 = 0$$

$$x^5 + 2x^3 + 4x^2 - 8x + 16 = 0$$

$a = 1,\ b = 0,\ c = 2,\ d = 4,\ e = -8,\ f = 16.$

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Complex numbers
Exercise I, Question 1

Question:

a Find the roots of the equation $z^2 + 2z + 17 = 0$ giving your answers in the form $a + \mathrm{i}b$, where a and b are integers.

b Show these roots on an Argand diagram.

Solution:

a

$$z^2 + 2z + 17 = 0$$
$$z^2 + 2z = -17$$
$$z^2 + 2z + 1 = -17 + 1 = -16$$

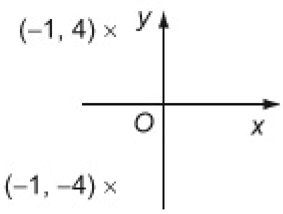
You may use any accurate method of solving a quadratic equation. Completing the square works well when the coefficient of z^2 is one and the coefficient of z is even.

$\sqrt{(-16)} = 4\sqrt{(-1)} = 4\mathrm{i}$

$$(z+1)^2 = -16$$
$$z + 1 = \pm 4\mathrm{i}$$

$z = -1 - 4\mathrm{i}, -1 + 4\mathrm{i}$

b



In the Argand diagram, you must place points representing conjugate complex numbers symmetrically about the real x-axis.

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Complex numbers
Exercise I, Question 2

Question:

$z_1 = -\mathrm{i}, z_2 = 1 + \mathrm{i}\sqrt{3}$

a Find the modulus of

i $z_1 z_2$

ii $\frac{z_1}{z_2}$.

b Find the argument of

i $z_1 z_2$

ii $\frac{z_1}{z_2}$.

Give your answers in radians as exact multiples of π.

Solution:

a i

$$\begin{aligned} z_1 z_2 &= -\mathrm{i}(1 + \mathrm{i}\sqrt{3}) \\ &= -\mathrm{i} + \sqrt{3} \\ &= \sqrt{3} - \mathrm{i} \end{aligned}$$

$-\mathrm{i} \times \mathrm{i}\sqrt{3} = -(-1)\sqrt{3} = \sqrt{3}$

$$\begin{aligned} |z_1 z_2|^2 &= (\sqrt{3})^2 + (-1)^2 = 3 + 1 = 4 \\ |z_1 z_2| &= 2 \end{aligned}$$

You find the modulus of complex numbers using the result that, if $z = a + \mathrm{i}b$, then $|z|^2 = a^2 + b^2$. This result is essentially the same as Pythagoras' Theorem and so is easy to remember.

ii

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{-\mathrm{i}}{1 + \mathrm{i}\sqrt{3}} \times \frac{1 - \mathrm{i}\sqrt{3}}{1 - \mathrm{i}\sqrt{3}} \\ &= \frac{-\mathrm{i} - \sqrt{3}}{1^2 + (\sqrt{3})^2} = -\frac{\sqrt{3}}{4} - \frac{1}{4}\mathrm{i} \end{aligned}$$

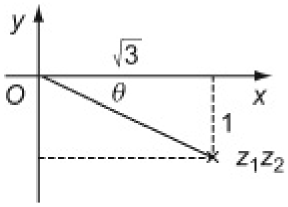
To simplify a quotient, you multiply the numerator and denominator by the conjugate complex of the denominator. The conjugate complex of this denominator, $1 + \mathrm{i}\sqrt{3}$, is $1 - \mathrm{i}\sqrt{3}$.

$$\left|\frac{z_1}{z_2}\right|^2 = \left(-\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{3}{16} + \frac{1}{16} = \frac{1}{4}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{1}{2}$$

b i

$z_1 z_2 = \sqrt{3} - \mathrm{i}$



$\tan \theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

$z_1 z_2$ is in the fourth quadrant.

$\arg (z_1 z_2) = -\frac{\pi}{6}$

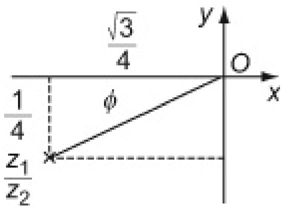
You draw a sketch of the Argand diagram to check which quadrant your complex number is in.

You usually work out an angle in a right angled triangle using a tangent.

You then adjust you angle to the correct quadrant. The argument is measured from the positive *x*-axis. This is clockwise and, hence, negative.

ii

$\frac{z_1}{z_2} = -\frac{\sqrt{3}}{4} - \frac{1}{4}\mathrm{i}$

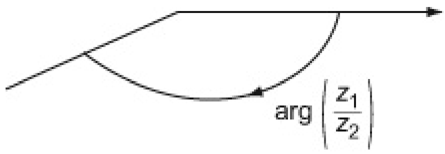


$\tan \phi = \frac{\frac{1}{4}}{\frac{\sqrt{3}}{4}} = \frac{1}{\sqrt{3}} \Rightarrow \phi = \frac{\pi}{6}$

$\frac{z_1}{z_2}$ is in the third quadrant.

$\arg\left(\frac{z_1}{z_2}\right) = -\left(\pi - \frac{\pi}{6}\right) - \frac{5\pi}{6}$

This complex number is in the third quadrant. Again the argument is negative.



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Edexcel AS and A Level Modular Mathematics

Complex numbers
Exercise I, Question 3

Question:

$z = \frac{1}{2+\mathrm{i}}$.

a Express in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$,

i z^2

ii $z - \frac{1}{z}$.

b Find $|z^2|$.

c Find $\arg\left(z - \frac{1}{z}\right)$, giving your answer in degrees to one decimal place.

Solution:

a i

$$\begin{aligned} z &= \frac{1}{2+\mathrm{i}} \times \frac{2-\mathrm{i}}{2-\mathrm{i}} = \frac{2-\mathrm{i}}{5} \\ &= \frac{2}{5} - \frac{1}{5}\mathrm{i} \end{aligned}$$

It is useful to be able to write down the product of a complex number and its conjugate without doing a lot of working. $(a+\mathrm{i}b)(a-\mathrm{i}b) = a^2 + b^2$ This is sometimes called the formula for the sum of two squares. It has a similar pattern to the formula for the difference of two squares. $(a+b)(a-b) = a^2 - b^2$

$$\begin{aligned} z^2 &= \left(\frac{2}{5} - \frac{1}{5}\mathrm{i}\right)^2 \\ &= \frac{4}{25} - \frac{4}{25}\mathrm{i} + \left(\frac{1}{5}\mathrm{i}\right)^2 \\ &= \frac{4}{25} - \frac{4}{25}\mathrm{i} - \frac{1}{25} \\ &= \frac{3}{25} - \frac{4}{25}\mathrm{i} \end{aligned}$$

You square using the formula $(a-b)^2 = a^2 - 2ab + b^2$

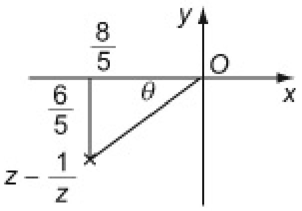
ii

$$\begin{aligned} z - \frac{1}{z} &= \frac{2}{5} - \frac{1}{5}\mathrm{i} - (2+\mathrm{i}) \\ &= \frac{2}{5} - \frac{1}{5}\mathrm{i} - 2 - \mathrm{i} \\ &= -\frac{8}{5} - \frac{6}{5}\mathrm{i} \end{aligned}$$

b

$$\begin{aligned}|z^2|^2 &= \left(\frac{3}{25}\right)^2 + \left(-\frac{4}{25}\right)^2 \\ &= \frac{9}{625} + \frac{16}{625} = \frac{25}{625} = \frac{1}{25} \\ |z^2| &= \frac{1}{5}\end{aligned}$$

c



You should draw a sketch to help you decide which quadrant the complex number is in.

$$\tan\theta = \frac{\frac{6}{5}}{\frac{8}{5}} = \frac{3}{4} \Rightarrow \theta \approx 36.87^\circ$$

$z - \frac{1}{z}$ is in the third quadrant

$$\begin{aligned}\arg\left(z - \frac{1}{z}\right) &= -(180^\circ - \theta) \\ &= -143.1^\circ, \text{to 1 d.p.}\end{aligned}$$

Arguments are measured from the positive x-axis. Angles measured clockwise are negative.

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Complex numbers
Exercise I, Question 4

Question:

The real and imaginary parts of the complex number $z = x + iy$ satisfy the equation $(2 - i)x - (1 + 3i)y - 7 = 0$.

a Find the value of x and the value of y.

b Find the values of

i $|z|$

ii arg z.

Solution:

a

$$2x - xi - y - 3yi - 7 = 0$$

You find two simultaneous equations by equating the real and imaginary parts of the equation.

$$(2x - y - 7) + (-x - 3y)i = 0 + 0i$$

You think of 0 as 0 + 0i, a number which has both its real and imaginary parts zero.

Equating real and imaginary parts

Real $2x - y - 7 = 0$

Imaginary $-x - 3y = 0$

$$2x - y = 7 \quad \textbf{(1)}$$
$$x + 3y = 0 \quad \textbf{(2)}$$
$$2 \times \textbf{(2)} \quad 2x + 6y = 0 \quad \textbf{(3)}$$
$$\textbf{(3)} - \textbf{(1)} \quad 7y = -7 \Rightarrow y = -1$$

The simultaneous equations are solved in exactly the same way as you learnt for GCSE.

Substitute into **(2)**

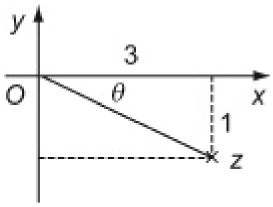
$$\begin{aligned} x - 3 &= 0 \Rightarrow x = 3 \\ x &= 3, y = -1 \end{aligned}$$

b i

$$\begin{aligned} z &= 3 - i \\ |z| &= 3^2 + (-1)^2 = 10 \\ |z| &= \sqrt{10} \end{aligned}$$

ii

As the question has not specified that you should work in radians or degrees, you could work in either and -18.4° would also be an



$\tan\theta = \frac{1}{3} \Rightarrow \theta \approx 0.322$, in radians

z is in the fourth quadrant.

$\arg\ z = -0.322$, in radians to 3 d.p.

acceptable answer.

The question did not specify any accuracy. 3 significant figures is a sensible accuracy but you could give more.

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Complex numbers
Exercise I, Question 5

Question:

Given that $2 + \mathrm{i}$ is a root of the equation $z^3 - 11z + 20 = 0$, find the other roots of the equation.

Solution:

One other root is $2 - \mathrm{i}$.

If $a + \mathrm{i}b$ is a root, then $a - \mathrm{i}b$ must also be a root. The complex roots of polynomials with real coefficients occur as complex conjugate pairs.

The cubic equation must be identical to

$$(z - 2 - \mathrm{i})(z - 2 + \mathrm{i})(z - \gamma) = 0$$

If α, β and γ are the roots of a cubic equation, then the equation must have the form $(x - \alpha)(x - \beta)(x - \gamma) = 0$.

$$((z - 2) - \mathrm{i})((z - 2) + \mathrm{i}) = (z - 2)^2 - \mathrm{i}^2$$

$$= z^2 - 4z + 4 + 1 = z^2 - 4z + 5$$

You know the first two roots, α and β, so the only remaining problem is finding the third root γ.

Hence

$$\left(z^2 - 4z + 5\right)(z - \gamma) = z^3 - 11z + 20$$

You need not multiply the brackets on the left hand side of this equation out fully. If the brackets were multiplied out, the only term without a z would be when $+5$ is multiplied by $-\gamma$ and the product of these, -5γ, equals the term without z on the right hand side, $+20$.

Equating constant coefficients

$$-5\gamma = 20 \Rightarrow \gamma = -4$$

The other roots are $2 - \mathrm{i}$ and -4.

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Complex numbers
Exercise I, Question 6

Question:

Given that $1 + 3\text{i}$ is a root of the equation $z^3 + 6z + 20 = 0$,

a find the other two roots of the equation,

b show, on a single Argand diagram, the three points representing the roots of the equation,

c prove that these three points are the vertices of a right-angled triangle.

Solution:

a One other root is $1 - 3\text{i}$

If $a + \text{i}b$ is a root, then $a - \text{i}b$ must also be a root. The complex roots of polynomials with real coefficients occur as complex conjugate pairs.

The cubic equation must be identical to
$(z - 1 - 3\text{i})(z - 1 + 3\text{i})(z - \gamma) = 0$

If α, β and γ are the roots of a cubic equation, then the equation must have the form $(x - \alpha)(x - \beta)(x - \gamma) = 0$.

$$((z-1) - 3\text{i})((z-1) + 3\text{i}) = (z-1)^2 - (3\text{i})^2$$
$$= z^2 - 2z + 1 + 9 = z^2 - 2z + 10$$

You know the first two roots, α and β, so the only remaining problem is finding the third γ.

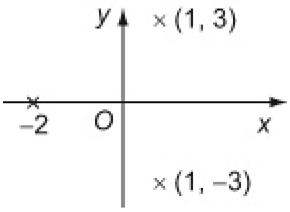
Hence
$\left(z^2 - 2z + 10\right)(z - \gamma) = z^3 + 6z + 20$

You need not multiply the brackets on the left hand side of this equation out fully. If the brackets were multiplied out, the only term without a z would be when $+10$ is multiplied by $-\gamma$ and the product of these, -10γ, equals the term without z on the right hand side, $+20$.

Equating constant coefficients
$-10\gamma = 20 \Rightarrow \gamma = -2$

The other roots are $1 - 3\text{i}$and -2.

b



c

The gradient of the line joining $(-2,0)$ to $(1,3)$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-0}{1-(-2)} = \frac{3}{3} = 1$$

You prove the result in part (c) using the methods of Coordinate Geometry that you learnt for the C1 module. These can be found in Edexcel Modular Mathematics for AS and A-level Core Mathematics 1, Chapter 5.

The gradient of the line joining $(-2,0)$ to $(1,-3)$ is given by

$$m' = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-0}{1-(-2)} = \frac{-3}{3} = -1$$

Hence $mm' = -1$, which is the condition for perpendicular lines.

Two sides of the triangle are at right angles to each other and the triangle is right-angled.

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Edexcel AS and A Level Modular Mathematics

Complex numbers
Exercise I, Question 7

Question:

$z_1 = 4 + 2i, z_2 = -3 + i$

a Display points representing z_1 and z_2 on the same Argand diagram.

b Find the exact value of $|z_1 - z_2|$.

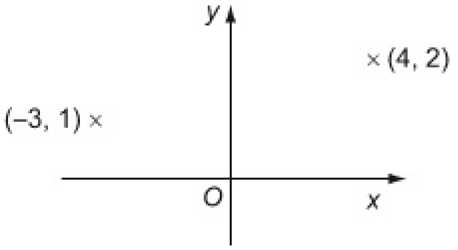
Given that $w = \frac{z_1}{z_2}$,

c express w in the form $a + ib$, where $a, b \varepsilon \mathbb{R}$,

d find arg w, giving your answer in radians.

Solution:

a



b

$$\begin{aligned} z_1 - z_2 &= 4 + 2i - (-3 + i) \\ &= 4 + 2i + 3 - i = 7 + i \end{aligned}$$

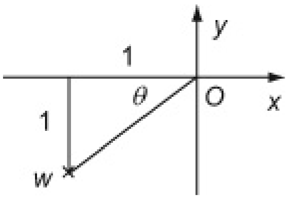
$z_1 - z_2$ could be represented by the vector joining the point (−3,1) to the point (4, 2). $|z_1 - z_2|$ is then the distance between these two points.

$$\begin{aligned} |z_1 - z_2|^2 &= 7^2 + 1^2 = 50 \\ |z_1 - z_2| &= \sqrt{50} = 5\sqrt{2} \end{aligned}$$

The question specifies an exact answer, so decimals would not be acceptable.

c

$$\begin{aligned} w &= \frac{4 + 2i}{-3 + i} \times \frac{-3 - i}{-3 - i} = \frac{-12 - 4i - 6i + 2}{(-3)^2 + 1^2} \\ &= \frac{-10 - 10i}{10} = -1 - i \end{aligned}$$



$\tan\ \theta = \frac{\frac{1}{4}}{\frac{1}{4}} = 1 \Rightarrow \theta = \frac{\pi}{4}$

w is in the third quadrant.

$\arg\ w = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$

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Edexcel AS and A Level Modular Mathematics

Complex numbers
Exercise I, Question 8

Question:

Given that $3-2\mathrm{i}$ is a solution of the equation

$x^4-6x^3+19x^2-36x+78=0$,

a solve the equation completely,

b show on a single Argand diagram the four points that represent the roots of the equation.

Solution:

a

Let $\mathrm{f}(x)=x^4-6x^3+19x^2-36x+78$

When you have to refer to a long expression, like this quartic equation, several times in a solution, it saves time to call the expression, say, $\mathrm{f}(x)$. It is much quicker to write $\mathrm{f}(x)$ than $x^4-6x^3+19x^2-36x+78$!

As $3-2\mathrm{i}$ is a root of $\mathrm{f}(x)$, $3+2\mathrm{i}$ is also a root of $\mathrm{f}(x)$.

$$\begin{aligned}(x-3+2\mathrm{i})(x-3+2\mathrm{i}) &= (x-3)^2+4\\ &= x^2-6x+9+4\\ &= x^2-6x+13\end{aligned}$$

If $a-\mathrm{i}\,b$ is a root, then $a+\mathrm{i}\,b$ must also be a root. The complex roots of polynomials with real coefficients occur as complex conjugate pairs.

$$\begin{array}{r|l} & \quad\quad\quad x^2+6 \\ \hline x^2-6x+13 & x^4-6x^3+19x^2-36x+78 \\ & \underline{x^4-6x^3+13x^2} \\ & \quad\quad\quad 6x^2-36x+78 \\ & \quad\quad\quad \underline{6x^2-36x+78} \end{array}$$

If α and β are roots of $\mathrm{f}(x)$, then $\mathrm{f}(x)$ must have the form $(x-\alpha)(x-\beta)\left(x^2+ax+b\right)$ and the remaining two roots can be found by solving $x^2+ax+b=0$. The method used here is finding a and b by long division. In this case $a=0$ and $b=6$.

Hence

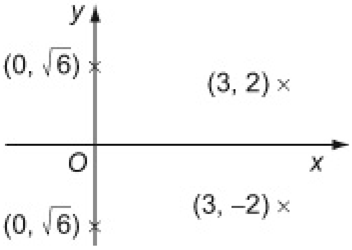
$$\mathrm{f}(x) = \left(x^2-6x+13\right)\left(x^2+6\right)=0$$

$$x^2+6=0 \;\Rightarrow\; x=\pm\mathrm{i}\sqrt{6}$$

The solutions of $\mathrm{f}(x)=0$ are

$$3-2\mathrm{i}, 3+2\mathrm{i}, \mathrm{i}\sqrt{6}, -\mathrm{i}\sqrt{6}$$

b


y
(0, √6)
(3, 2)
O
x
(3, –2)
(0, √6)

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Complex numbers
Exercise I, Question 9

Question:

$z = \frac{a+3\text{i}}{2+a\text{i}}, \quad a \,\varepsilon\, \mathbb{R}.$

a Given that $a = 4$, find $|z|$.

b Show that there is only one value of a for which $\arg z = \frac{\pi}{4}$, and find this value.

Solution:

a

$$\begin{aligned} z &= \frac{a+3\text{i}}{2+a\text{i}} = \frac{a+3\text{i}}{2+a\text{i}} \times \frac{2-a\text{i}}{2-a\text{i}} \\ &= \frac{2a-a^2\text{i}+6\text{i}+3a}{4+a^2} \\ &= \frac{5a}{4+a^2} + \frac{6-a^2}{4+a^2}\text{i} \ldots\ldots * \end{aligned}$$

You could substitute $a = 4$ into the expression for z at the beginning of part (a) and this would actually make this part easier. However you can use the expression marked * once in this part and three times in part (b) as well. It often pays to read quickly right through a question before starting.

Substitute $a = 4$

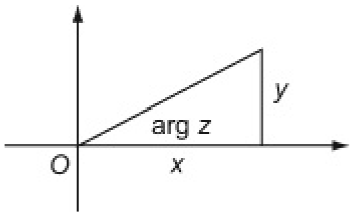
$z = \frac{20}{20} + \frac{-10}{20}\text{i} = 1 - \frac{1}{2}\text{i}$

$|z|^2 = 1^2 + \left(-\frac{1}{2}\right)^2 = \frac{5}{4}$

$|z| = \frac{\sqrt{5}}{2}$

b

$\tan(\arg z) = \frac{\frac{5a}{4+a^2}}{\frac{6-a^2}{4+a^2}} = \frac{5a}{6-a^2}$



If $z = x + \text{i}\,y$, then $\tan(\arg z) = \frac{y}{x}$.

Also from the data in the question

$\tan(\arg z) = \tan\frac{\pi}{4} = 1$

Hence

$\frac{5a}{6-a^2} = 1 \Rightarrow 5a = 6 - a^2 \Rightarrow a^2 + 5a - 6 = 0$

$(a-1)(a+6) = 0 \Rightarrow a = 1\,, -6$

If $a = -6$, substituting into the result $*$ in part (a)

$z = \frac{30}{40} - \frac{30}{40}\text{i} = \frac{3}{4} - \frac{3}{4}\text{i}$

This is in the third quadrant and has a negative

At this point you have two answers. The question asks you to show that there is only one value of a. You must test both and choose the one that satisfies the condition $\arg z = \frac{\pi}{4}$. The other value occurs because argument $\left(-\frac{3\pi}{4}\right)$, so $a = -6$ is rejected.

$\tan\frac{\pi}{4}$ and $\tan\left(-\frac{3\pi}{4}\right)$ are both 1.

If $a = 1$, substituting into the result $*$ in part (a)

$z = \frac{5}{5} + \frac{5}{5}\mathrm{i} = 1 + \mathrm{i}$

This is in the first quadrant and does have an argument $\frac{\pi}{4}$.

$a = 1$ is the only possible value of a.

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Numerical solutions of equations
Exercise A, Question 1

Question:

Use interval bisection to find the positive square root of $x^2 - 7 = 0$, correct to one decimal place.

Solution:

$x^2 - 7 = 0$

So roots lies between 2 and 3 as $f(2) = -3$ and $f(2) = +$ Using table method.

a	$f(a)$	b	$f(b)$	$\frac{a+b}{2}$	$\frac{f(a+b)}{2}$
2	−3	3	+2	2.5	−0.75
2.5	−0.75	3	+2	2.75	0.5625
2.5	−0.75	2.75	0.5625	2.625	−0.109375
2.625	−0.109375	2.75	0.5625	2.6875	0.2226562
2.625	−0.109375	2.6875	0.2226562	2.65625	0.055664
2.625	−0.109375	2.65625	0.055664	2.640625	−0.0270996

Hence $x^2 - 7 = 0$ when $x = 2.6$ to 1decimal place

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Numerical solutions of equations
Exercise A, Question 2

Question:

a Show that one root of the equation $x^3 - 7x + 2 = 0$ lies in the interval [2, 3].

b Use interval bisection to find the root correct to two decimal places.

Solution:

a $f(2) = 8 - 14 + 2 = -4$ $\quad f(x) = x^3 - 7x + 2$

$f(3) = 27 - 21 + 2 = +8$

Hence change of sign, implies roots between 2 and 3.

b Using table method.

a	$f(a)$	b	$f(b)$	$\frac{a+b}{2}$	$\frac{f(a+b)}{2}$
2	−4	3	+8	2.5	0.125
2	−4	2.5	0.125	2.25	−2.359375
2.25	−2.359375	2.5	0.125	2.375	−1.2285156
2.375	−1.2285156	2.5	0.125	2.4375	−0.5803222
2.4375	−0.5803222	2.5	0.125	2.46875	−0.2348938
2.46875	−0.2348938	2.5	0.125	2.484375	−0.0567665
2.484375	−0.0567665	2.5	0.125	2.4921875	0.0336604
2.484375	−0.0567665	2.4921875	0.0336604	2.4882813	−0.0116673

Hence $x = 2.49$ to 2 decimal places.

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Numerical solutions of equations
Exercise A, Question 3

Question:

a Show that the largest positive root of the equation $0 = x^3 + 2x^2 - 8x - 3$ lies in the interval [2, 3].

b Use interval bisection to find this root correct to one decimal place.

Solution:

a $f(2) = 8 + 8 - 16 - 3 = -3$ $\quad f(x) = x^3 + 2x^2 - 8x - 3$

$f(3) = 27 + 18 - 24 - 3 = 18$

Change of sign implies root in interval [2,3]

b

a	$f(a)$	b	$f(b)$	$\frac{a+b}{2}$	$f\left(\frac{a+b}{2}\right)$
2	−3	3	18	2.5	5.125
2	−3	2.5	5.125	2.25	0.51562
2	−3	2.25	0.515625	2.125	−1.37304
2.125	−1.3730469	2.25	0.515625	2.1875	−0.46215

Hence solution = 2.2 to 1decimal place

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Numerical solutions of equations
Exercise A, Question 4

Question:

a Show that the equation $f(x) = 1 - 2\sin x$ has one root which lies in the interval [0.5, 0.8].

b Use interval bisection four times to find this root. Give your answer correct to one decimal place.

Solution:

a $f(0.5) = +0.0411489$

$f(0.8) = -0.4347121$

Change of sign implies root between 0.5 and 0.8

b

a	$f(a)$	b	$f(b)$	$\frac{a+b}{2}$	$\frac{f(a+b)}{2}$
0.5	0.0411489	0.8	−0.4347121	0.65	−0.2103728
0.5	0.0411489	0.65	−0.2103728	0.575	−0.0876695
0.5	0.0411489	0.575	−0.0876696	0.5375	−0.0239802

0.5 to 1 decimal place.

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Numerical solutions of equations
Exercise A, Question 5

Question:

a Show that the equation $0 = \frac{x}{2} - \frac{1}{x},\ x > 0$, has a root in the interval [1, 2].

b Obtain the root, using interval bisection two times. Give your answer to two significant figures.

Solution:

a $f(1) = -0.5 \qquad p = \frac{1}{2} + x - \frac{1}{x}$

$f(2) = +0.5$

Change of sign implies root between interval [1,2]

b

a	$f(a)$	b	$f(b)$	$\frac{a+b}{2}$	$f\left(\frac{a+b}{2}\right)$
1	−0.5	2	+0.5	1.5	0.0833
1	−0.5	1.5	0.083	1.25	−0.175
1.25	−0.175	1.5	0.083	1.375	−0.0397727
1.375	−0.0397727	1.5	0.083	1.4375	0.0230978

Hence $x = 1.4$ to 2 significant figures

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Numerical solutions of equations
Exercise A, Question 6

Question:

$f(x) = 6x - 3^x$

The equation $f(x) = 0$ has a root between $x = 2$ and $x = 3$. Starting with the interval [2, 3] use interval bisection three times to give an approximation to this root.

Solution:

a	$f(a)$	b	$f(b)$	$\frac{a+b}{2}$	$f\left(\frac{a+b}{2}\right)$
2	3	3	−9	2.5	−0.588457
2	3	2.5	−0.5884572	2.25	1.65533
2.25	1.6553339	2.5	−0.5884572	2.375	0.66176
2.375	0.6617671	2.5	−0.5884572	2.4375	0.0708
2.4375	0.0709769	2.5	−0.5844572	2.46875	−0.2498
2.4375	0.0709769	2.46875	−0.2498625	2.453125	−0.08726
2.4375	0.0709769	2.453125	−0.0872613	2.4453125	−0.0076

2.4 correct to 1 decimal place.

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Numerical solutions of equations
Exercise B, Question 1

Question:

a Show that a root of the equation $x^3 - 3x - 5 = 0$ lies in the interval [2, 3].

b Find this root using linear interpolation correct to one decimal place.

Solution:

a $f(2) = 8 - 6 - 5 = -3 \quad f(x) = x^3 - 3x - 5$

$f(3) = 27 - 9 - 5 = +13$

Change of size therefore root in interval [2, 3]

b Using linear interpolation and similar triangle taking x_1 as the first root.

$$\frac{3 - x_1}{x_1 - 2} = \frac{3}{13} \quad x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

so

$$\begin{aligned} 13(3 - x_1) &= 3(x_1 - 2) \\ 39 - 13x_1 &= 3x_1 - 6 \\ 16x_1 &= 45 \\ x_1 &= 2.8125 \quad f(x_1) = 8.8098 \end{aligned}$$

Using interval (2, 2.8125)

$$\begin{aligned} \frac{2.8125 - x_2}{x_2 - 2} &= \frac{3}{8.8098} \\ x_2 &= 2.606 \quad f(x_2) = 4.880 \end{aligned}$$

Using interval (2, 2.606)

$$\begin{aligned} \frac{2.606 - x_3}{x_3 - 2} &= \frac{3}{4.880} \\ x_2 &= 2.375 \quad f(x_2) = 1.276 \end{aligned}$$

Using interval (2, 2.375)

$$\begin{aligned} \frac{2.375 - x_4}{x_4 - 2} &= \frac{3}{1.276} \\ x_2 &= 2.112 \quad f(x_4) = -1.915 \end{aligned}$$

Using interval (2.112, 2.375)

$$\begin{aligned} \frac{2.375 - x_5}{x_5 - 2.112} &= \frac{1.915}{1.276} \\ &= 2.218 \quad f(x_5) = -0.736 \end{aligned}$$

Using interval (2.218, 2.375)

$$\frac{2.375 - x_6}{x_6 - 2.218} = \frac{0.736}{1.276}$$

$= 2.318 \quad f(x_6) = 0.494$

Using interval (2.218, 2.318)

$$\frac{2.318 - x_7}{x_7 - 2.218} = \frac{0.736}{0.494}$$

$= 2.25 \quad f(x_7) = -0.229$

2.3 to 1 decimal place.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Numerical solutions of equations
Exercise B, Question 2

Question:

a Show that a root of the equation $5x^3 - 8x^2 + 1 = 0$ has a root between $x = 1$ and $x = 2$.

b Find this root using linear interpolation correct to one decimal place.

Solution:

a $f(1) = 5 - 8 + 1 = -2 \quad f(x) = 5x^3 - 8x^2 + 1$

$f(2) = 40 - 32 + 1 = +9$

Therefore root in interval [1, 2] as sign change.

b Using linear interpolation.

$$\frac{2 - x_1}{x_1 - 1} = \frac{2}{9}$$
$$x_1 = 1.818 \quad f(x_1) = 4.612.$$

Using interval (1, 1.818)

$$\frac{1.818 - x_2}{x_2 - 1} = \frac{2}{4.612}$$
$$x_2 = 1.570 \quad f(x_2) = 0.647$$

Using interval (1, 1.570)

$$\frac{1.570 - x_3}{x_3 - 1} = \frac{2}{0.647}$$
$$x_3 = 1.139 \quad f(x_3) = -1.984$$

Using interval (1.139, 1.570)

$$\frac{1.570 - x_4}{x_4 - 1.139} = \frac{1.984}{0.647}$$
$$x_4 = 1.447 \quad f(x_4) = -0.590$$

Use interval (1.447, 1.570)

$$\frac{1.570 - x_5}{x_5 - 1.447} = \frac{0.590}{0.647}$$
$$= 1.511 \quad f(x_5) = -0.0005.$$

Ans 1.5 correct to 1 decimal place.

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Numerical solutions of equations
Exercise B, Question 3

Question:

a Show that a root of the equation $\frac{3}{x}+3=x$ lies in the interval [3, 4].

b Use linear interpolation to find this root correct to one decimal place.

Solution:

a $f(3)=1 \quad f(x)=\frac{3}{x}+3-x$

$f(4)=-0.25$

Hence root as sign change in interval [3, 4]

b Using linear interpolation

$$\frac{4-x_1}{x_1-3} = \frac{0.25}{1}$$
$$x_1 = 3.8 \quad f(x_1)=-0.011$$

Using interval [3, 3.8]

$$\frac{3.8-x_2}{x_2-3} = \frac{0.0111}{1}$$
$$x_2 = 3.791 \quad f(x_2)=-0.0004579$$

Ans = 3.8 to 1decimal place

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Numerical solutions of equations
Exercise B, Question 4

Question:

a Show that a root of the equation $2x \cos x - 1 = 0$ lies in the interval [1, 1.5].

b Find this root using linear interpolation correct to two decimal places.

Solution:

a f(1) = 0.0806

f(1.5) = −0.788

Hence root between (1, 1.5) as sign change

b Using linear interpolation

$$\frac{1.5 - x_1}{x_1 - 1} = \frac{0.788}{1}$$
$$x_1 = 1.280 \quad f(1.280) = -0.265$$

Use interval [1, 1.28]

$$\frac{1.28 - x_2}{x_2 - 1} = \frac{0.265}{1}$$
$$x_2 = 1.221 \quad f(1.221) = -0.164$$

Use interval [1, 1.221]

$$\frac{1.221 - x_2}{x_3 - 1} = \frac{0.164}{1}$$
$$x_3 = 1.190 \quad f(1.190) = -0.115$$

Use interval [1, 1.190]

$$\frac{1.190 - x_4}{x_4 - 1} = \frac{0.115}{1}$$
$$x_4 = 1.170 \quad f(1.170) = 0.088$$

Use interval [1, 1.170]

$$\frac{1.170 - x_5}{x_5 - 1} = \frac{0.088}{1}$$
$$x_5 = 1.156 \quad f(1.156) = -0.068$$

Root 1.10 to 2 decimal places.

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Numerical solutions of equations
Exercise B, Question 5

Question:

a Show that the largest possible root of the equation $x^3 - 2x^2 - 3 = 0$ lies in the interval [2, 3].

b Find this root correct to one decimal place using interval interpolation.

Solution:

a $f(2) = 8 - 8 - 3 = -3 \quad f(x) = x^3 - 2x^2 - 3$

$f(3) = 27 - 18 - 3 = 6$

Hence root lies in interval [2, 3] and $\forall\, x \in \; x \geq 3 f(x) < 0$.

b Using linear interpolation

$$\frac{3 - x_1}{x_1 - 2} = \frac{6}{3}$$
$$x_1 = 2.333 \quad f(x_1) = -1.185$$

$$\frac{3 - x_2}{x_2 - 2.333} = \frac{6}{1.185}$$
$$x_2 = 2.443 \quad f(x_2) = -0.356$$

$$\frac{3 - x_3}{x_3 - 2.443} = \frac{6}{0.356}$$
$$x_3 = 2.474 \quad f(x_3) = -0.095$$

$$\frac{3 - x_4}{x_4 - 2.474} = \frac{6}{0.095}$$
$$x_4 = 2.482$$

Hence root = 2.5 to 1 d.p

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Numerical solutions of equations
Exercise B, Question 6

Question:

$f(x) = 2^x - 3x - 1$

The equation $f(x) = 0$ has a root in the interval [3, 4].

Using this interval find an approximation to x.

Solution:

Let root be α

$$f(3) = -2$$
$$f(4) = 3$$

$$\frac{4-\alpha}{\alpha-3} = \frac{3}{2}$$
$$\alpha = 3.4 \quad \text{is the approximation.}$$

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Numerical solutions of equations
Exercise C, Question 1

Question:

Show that the equation $x^3 - 2x - 1 = 0$ has a root between 1 and 2. Find the root correct to two decimal places using the Newton–Raphson process.

Solution:

$f(1) = -2 \qquad f(x) = x^3 - 2x - 1$
$f(2) = 3 \qquad f(2) = 3$ is correct

Hence root in interval [1,2] as sign change

$$\begin{aligned} f(x) &= x^3 - 2x - 1 \\ f'(x) &= 3x^2 - 2 \end{aligned}$$

Let $x_0 = 2$.

$$\begin{aligned} \text{Then } x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ x_1 &= 2 - \frac{3}{10} \\ x_1 &= 1.7 \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ x_2 &= 1.88 - \frac{1.885}{8.6032} \\ &= 1.661 \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ x_3 &= 1.661 - \frac{0.2597}{6.2767} \\ &= 1.6120 \\ x_4 &= 1.620 - \frac{f(1.620)}{f'(1.620)} \\ x_4 &= 1.62 - \frac{0.0115}{5.8732} \\ &= 1.618 \\ \text{Solution } &= 1.62 \text{ to } 2 \text{ decimal places} \end{aligned}$$

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Edexcel AS and A Level Modular Mathematics

Numerical solutions of equations
Exercise C, Question 2

Question:

Use the Newton–Raphson process to find the positive root of the equation $x^3 + 2x^2 - 6x - 3 = 0$ correct to two decimal places.

Solution:

$$\begin{aligned} f(0) &= -3 \qquad f(x) = x^3 + 2x^2 - 6x - 3 \\ f(1) &= 1 + 2 - 6 - 3 = -6 \\ f(2) &= 8 + 8 - 12 - 3 = 1 \end{aligned}$$

Hence root in interval [1,2]

Using Newton Raphson

$$\begin{aligned} f(x) &= x^3 + 2x^2 - 6x - 3 \\ f'(x) &= 3x^2 + 4x - 6 \\ x_0 &= 2 \end{aligned}$$

$$\begin{aligned} \text{Then } x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 2 - \frac{1}{14} \\ &= 1.92857 \\ x_2 &= 1.92857 - \frac{0.0404494}{12.872427} \\ &= 1.92857 - 0.00314 \\ &= 1.9254 \end{aligned}$$

Root = 1.93 to 2 decimal places.

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Numerical solutions of equations
Exercise C, Question 3

Question:

Find the smallest positive root of the equation $x^4 + x^2 - 80 = 0$ correct to two decimal places. Use the Newton–Raphson process.

Solution:

$$\begin{aligned} f(x) &= x^4 + x^2 - 80 \\ f'(x) &= 4x^3 + 2x \end{aligned}$$

$$\begin{aligned} \text{Let } x_0 &= 3 \qquad f(3) = 10 \\ \text{So } x_1 &= 3 - \frac{f(x_0)}{f'(x_0)} \\ x_1 &= 3 - \frac{10}{114} \\ &= 2.912 \\ \text{Then } x_2 &= 2.912 - \frac{0.1768}{104.388} \\ &= 2.908 \end{aligned}$$

Hence root = 2.91 to 2 decimal places.

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Numerical solutions of equations
Exercise C, Question 4

Question:

Apply the Newton–Raphson process to find the negative root of the equation $x^3 - 5x + 2 = 0$ correct to two decimal places.

Solution:

$$\begin{aligned} f(x) &= x^3 - 5x + 2 \\ f'(x) &= 3x^2 - 5 \end{aligned}$$

$$\begin{aligned} f(0) &= 2 \\ f(-1) &= -1 + 5 + 2 = 6 \\ f(-2) &= -8 + 10 + 2 = 4 \\ f(-3) &= -27 + 15 + 2 = -10 \end{aligned}$$

Hence root between interval $[-2,-3]$

Let $x_0 = -2$

$$\begin{aligned} \text{Then } x_1 &= -2 - \frac{f(x_0)}{f'(x_0)} \\ &= -2 - \frac{4}{7} \\ &= -2.5714 \\ x_2 &= -2.571 - \frac{f(x_1)}{f'(x_1)} \\ &= -2.571 - \frac{2.1394}{14.83} \\ &= -2.4267 \\ x_3 &= -2.4267 - \frac{0.1570}{12.6662} \\ &= -2.4267 - 0.01234 \\ &= -2.439 \\ x_4 &= -2.439 - \frac{0.00163}{12.846} \\ &= -2.4391 \end{aligned}$$

Root $= -2.44$ correct to 2 decimal places.

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Numerical solutions of equations
Exercise C, Question 5

Question:

Show that the equation $2x^3 - 4x^2 - 1 = 0$ has a root in the interval [2, 3]. Taking 3 as a first approximation to this root, use the Newton–Raphson process to find this root correct to two decimal places.

Solution:

$$\begin{aligned} f(x) &= 2x^3 - 4x^2 - 1. \\ f(2) &= 16 - 16 - 1 = -1 \\ f(3) &= 54 - 36 - 1 = 17 \end{aligned}$$

Sign change implies root in interval [2,3]

$f'(x) = 6x^2 - 8x$

Let $x_0 = 3$

$$\begin{aligned} \text{Then}\quad x_1 &= 3 - \frac{f(x_0)}{f'(x_0)} \\ &= 3 - \frac{17}{30} \\ &= 2.43 \\ x_2 &= 2.43 - \frac{f(2.43)}{f'(2.43)} \\ &= 2.43 - \frac{4.078}{16.05} \\ &= 2.43 - 0.254 \\ &= 2.179 \\ x_3 &= 2.179 - \frac{f(2.179)}{f'(2.179)} \\ &= 2.179 - \frac{0.6998}{11.056} \\ &= 2.179 - 0.063296 \\ &= 2.116 \\ x_4 &= 2.116 - \frac{f(2.116)}{f'(2.116)} \\ &= 2.116 - \frac{0.0388}{9.937} = 2.112 \\ x_5 &= 2.112 - \frac{f(2.112)}{f'(2.112)} \\ &= 2.112 - \frac{-0.00084}{9.8672} \\ &= 2.112 \end{aligned}$$

Ans = 2.11 correct to 2 decimal place.

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Numerical solutions of equations
Exercise C, Question 6

Question:

$\mathrm{f}(x) = x^3 - 3x^2 + 5x - 4$

Taking 1.4 as a first approximation to a root, x, of this equation, use Newton–Raphson process once to obtain a second approximation to x. Give your answer to three decimal places.

Solution:

$$\begin{aligned} \mathrm{f}(x) &= x^3 - 3x^2 + 5x - 4 \\ \mathrm{f}'(x) &= 3x^2 - 6x + 5 \end{aligned}$$

Let $x_0 = 1.4$

Using Newton Raphson

$$\begin{aligned} x_1 &= 1.4 - \frac{\mathrm{f}(1.4)}{\mathrm{f}'(1.4)} \\ &= 1.4 - \frac{-0.136}{2.48} \\ &= 1.4 + 0.0548 \\ &= 1.455 \text{ to } 3 \text{ decimal places} \end{aligned}$$

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Numerical solutions of equations
Exercise C, Question 7

Question:

Use the Newton–Raphson process twice to find the root of the equation $2x^3 + 5x = 70$ which is near to $x = 3$. Give your answer to three decimal places.

Solution:

$$\begin{aligned} f(x) &= 2x^3 + 5x - 70 \\ f'(x) &= 6x^2 + 5 \end{aligned}$$

Let $x_0 = 3$

Using Newton Raphson

$$\begin{aligned} x_1 &= 3 - \frac{f(3)}{f'(3)} \\ &= 3 - \frac{-1}{59} \\ &= 3.02 \\ x_2 &= 3.02 - \frac{f(3.02)}{f'(3.02)} \\ &= 3.02 - \frac{0.1872}{59.72} \\ &-3.017 \text{ to } 3 \text{ decimal places.} \end{aligned}$$

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Numerical solutions of equations
Exercise D, Question 1

Question:

Given that $f(x) = x^3 - 2x + 2$ has a root in the interval $[-1\ ,\ -2]$, use interval bisection on the interval $[-1\ ,\ -2]$ to obtain the root correct to one decimal place.

Solution:

$$\begin{aligned} f(x) &= x^3 - 2x + 2 \\ f(-1) &= -1 + 2 + 2 = +3 \\ f(-2) &= -8 + 4 + 2 = -2 \end{aligned}$$

Hence root in interval [−1, −2] as sign change

a	$f(a)$	b	$f(b)$	$\frac{a+b}{2}$	$\frac{f(a+b)}{2}$
−1	+3	−2	−2	−1.5	+1.625
−1.5	1.625	−2	−2	−1.75	0.141
−1.75	0.141	−2	−2	−1.875	−0.842
−1.75	0.141	−1.875	−0.841	−1.8125	−0.329
−1.75	0.141	−1.8125	−0.329	−1.78125	

Hence solution is −1.8 to 1 decimal place.

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Numerical solutions of equations
Exercise D, Question 2

Question:

Show that the equation $x^3 - 12x - 7.2 = 0$ has one positive and two negative roots. Obtain the positive root correct to three significant figures using the Newton–Raphson process.

Solution:

$f(x) = x^3 - 12x - 7.2 = 0$

$f(0) = -7.2$	$f(-1) = 3.8$
$f(1) = -18.2$	$f(-2) = 8.8$
$f(2) = -23.2$	$f(-3) = 1.8$
$f(3) = -16.2$	$f(-4) = -23.2$
$f(4) = 8.8$	

positive root between [3, 4]

negative roots between [0, −1], [−3, −4] Let $x_0 = 4$

Using $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

where $f(x) = x^3 - 12x - 7.2$

$f'(x) = 3x^2 - 12$

So $x_1 = 4 - \frac{8.8}{36}$

$$\begin{aligned}
x_1 &= 3.756 \text{ to 3d.p.} \\
x_2 &= 3.756 - \frac{0.716}{30.322} \\
x_2 &= 3.732 \\
x_3 &= 3.732 - \frac{0.011}{30.323} \\
x_3 &= 3.7316
\end{aligned}$$

Hence root = 3.73 to 3 significant figures

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Numerical solutions of equations
Exercise D, Question 3

Question:

Find, correct to one decimal place, the real root of $x^3 + 2x - 1 = 0$ by using the Newton–Raphson process.

Solution:

$$\begin{aligned} f(x) &= x^3 + 2x - 1 \\ f(0) &= -1 \\ f(1) &= 2 \end{aligned}$$

Hence root interval [0, 1]

Using $f(x) = x^3 + 2x - 1$

$$\begin{aligned} f'(x) &= 3x^2 + 2 \quad \text{and} \quad x_0 = 1 \\ x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ x_1 &= 1 - \frac{2}{5} \\ x_1 &= 0.6 \\ x_2 &= 0.6 - \frac{0.416}{3.08} \\ x_2 &= 0.465 \\ x_3 &= 0.465 - \frac{0.031}{2.647} \\ x_3 &= 0.453 \end{aligned}$$

Hence root is 0.5 to 1decimal place.

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Numerical solutions of equations
Exercise D, Question 4

Question:

Use the Newton–Raphson process to find the real root of the equation $x^3 + 2x^2 + 4x - 6 = 0$, taking $x = 0.9$ as the first approximation and carrying out one iteration.

Solution:

$$\begin{aligned}
\mathrm{f}(x) &= x^3 + 2x^2 + 4x - 6 \\
\mathrm{f}'(x) &= 3x^2 + 4x + 4 \\
\mathrm{f}(0.9) &= -0.051 \\
\mathrm{f}'(0.9) &= 10.03 \\
x_1 &= x_0 - \frac{\mathrm{f}(x_0)}{\mathrm{f}'(x_1)} \\
&= 0.9 - \frac{-0.051}{10.03} \\
&= 0.905 \text{ to 3 decimal places}
\end{aligned}$$

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Numerical solutions of equations
Exercise D, Question 5

Question:

Use linear interpolation to find the positive root of the equation $x^3 - 5x + 3 = 0$ correct to one decimal place.

Solution:

$$\begin{aligned} f(x) &= x^3 - 5x + 3 \\ f(1) &= -1 \\ f(2) &= +1. \end{aligned}$$

Hence positive root in interval [1, 2] Using linear interpolation and x, as the 1st approximation

$$\begin{aligned} \frac{2 - x_1}{x_1 - 1} &= \frac{1}{1} \\ 2 - x_1 &= x_1 - 1 \\ 2x_1 &= 3 \\ x_1 &= 1.5 \qquad f(x_1) = 1.125 \end{aligned}$$

Then

$$\begin{aligned} \frac{2 - x_2}{x_2 - 1.5} &= \frac{1}{1.125} \\ x_2 &= 1.882 \qquad f(x_2) = 0.260 \\ \frac{1.882 - x_2}{x_2 - 1.5} &= \frac{0.260}{1.125} \\ x_2 &= 1.810 \qquad f(x_3) = -0.117 \\ \frac{1.882 - x_4}{x_2 - 1.810} &= \frac{0.260}{0.117} \\ &= 1.832 \end{aligned}$$

root = 1.8 to 1 decimal place

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Numerical solutions of equations
Exercise D, Question 6

Question:

$f(x) = x^3 + x^2 - 6.$

a Show that the real root of $f(x) = 0$ lies in the interval [1, 2].

b Use the linear interpolation on the interval [1, 2] to find the first approximation to x.

c Use the Newton–Raphson process on $f(x)$ once, starting with your answer to **b**, to find another approximation to x, giving your answer correct to two decimal places.

Solution:

a

$$\begin{aligned} f(x) &= x^3 + x^2 - 6 \\ f(1) &= -4 \\ f(2) &= 6 \end{aligned}$$

Hence root in interval [1, 2]

b

$$\begin{aligned} \frac{2 - x_1}{x_1 - 1} &= \frac{6}{4} \\ x_1 &= 1.4 \end{aligned}$$

c

$$\begin{aligned} x_0 &= 1.4 \\ f(x) &= x^3 + x^2 - 6 \\ f'(x) &= 3x^2 + 2x \\ x_1 &= x_0 - \frac{f(x_0)}{f'(x_1)} \\ &= 1.4 - \frac{-1.296}{8.68} \\ &= 1.55 \text{ to 2 decimal places} \end{aligned}$$

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Numerical solutions of equations
Exercise D, Question 7

Question:

The equation $\cos x = \frac{1}{4}x$ has a root in the interval [1.0, 1.4]. Use linear interpolation once in the interval [1.0, 1.4] to find an estimate of the root, giving your answer correct to two decimal places.

Solution:

$\cos x = \frac{1}{4}x \Rightarrow \text{f}(x) = \frac{1}{4}x - \cos x$

$\text{f}(1) = -0.29$

$\text{f}(1.4) = 0.180$

$\frac{1.4 - x_1}{x_1 - 1} = \frac{-0.290}{-0.180}$

$x_1 = 1.153$

$x_1 = 1.15$ to 2 decimal places

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Numerical solutions of equations
Exercise D, Question 8

Question:

$f(x) = x^3 - 3x - 6$

Use the Newton–Raphson process to find the positive root of this equation correct to two decimal places.

Solution:

$$\begin{aligned} f(x) &= x^3 - 3x - 6 \\ f'(x) &= 3x^2 - 3 \end{aligned}$$

$$\begin{aligned} f(0) &= -5 & f(1) &= -7 \\ f(2) &= -3 & f(3) &= +13 \end{aligned}$$

Hence root in interval [2, 3]

Let $x_0 = 2$

Then

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_1)} \\ &= 2 - \frac{-3}{9} \\ x_1 &= 2.333 \\ x_2 &= -\frac{4.301}{16.500} \\ x_2 &= 2.297 \\ x_3 &= 2.297 - \frac{0.228}{12.828} \\ x_3 &= 2.279 \\ x_4 &= 2.279 - \frac{-0.000236}{12.582} \\ &= 2.279 + 0.000019 \\ x_4 &= 2.2790 \end{aligned}$$

Ans = 2.28 to 2 decimal places

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Quadratic Equations
Exercise A, Question 1

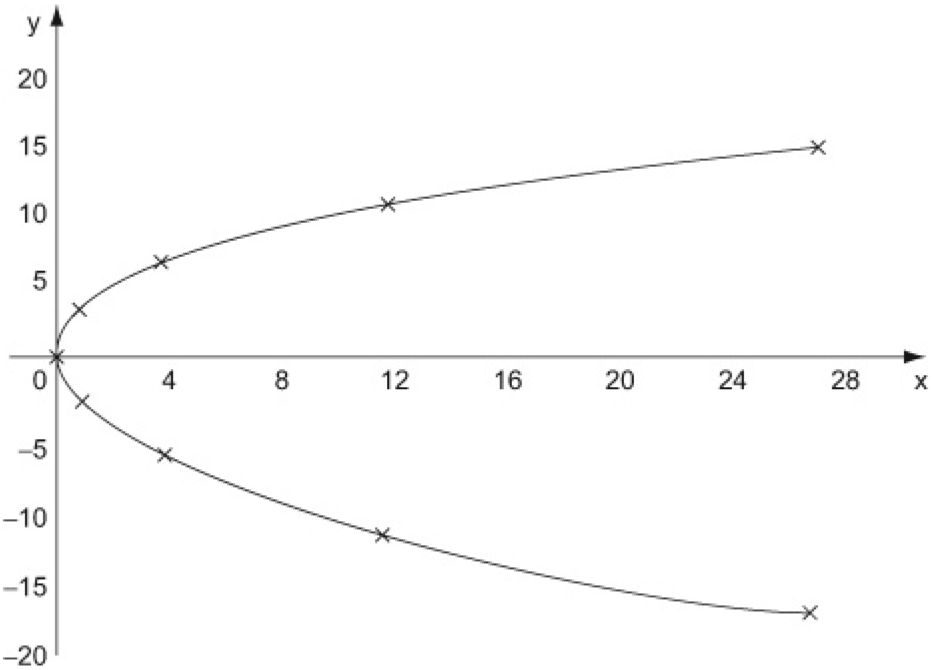
Question:

A curve is given by the parametric equations $x = 2t^2$, $y = 4t$. $t \in \mathbb{R}$. Copy and complete the following table and draw a graph of the curve for $-4 \le t \le 4$.

t	−4	−3	−2	−1	−0.5	0	0.5	1	2	3	4
$\boldsymbol{x = 2t^2}$	32					0	0.5				32
$\boldsymbol{y = 4t}$	−16						2				16

Solution:

t	−4	−3	−2	−1	−0.5	0	0.5	1	2	3	4
$x = 2t^2$	32	18	8	2	0.5	0	0.5	2	8	18	32
$y = 4t$	−16	−12	−8	−4	−2	0	2	4	8	12	16



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Quadratic Equations
Exercise A, Question 2

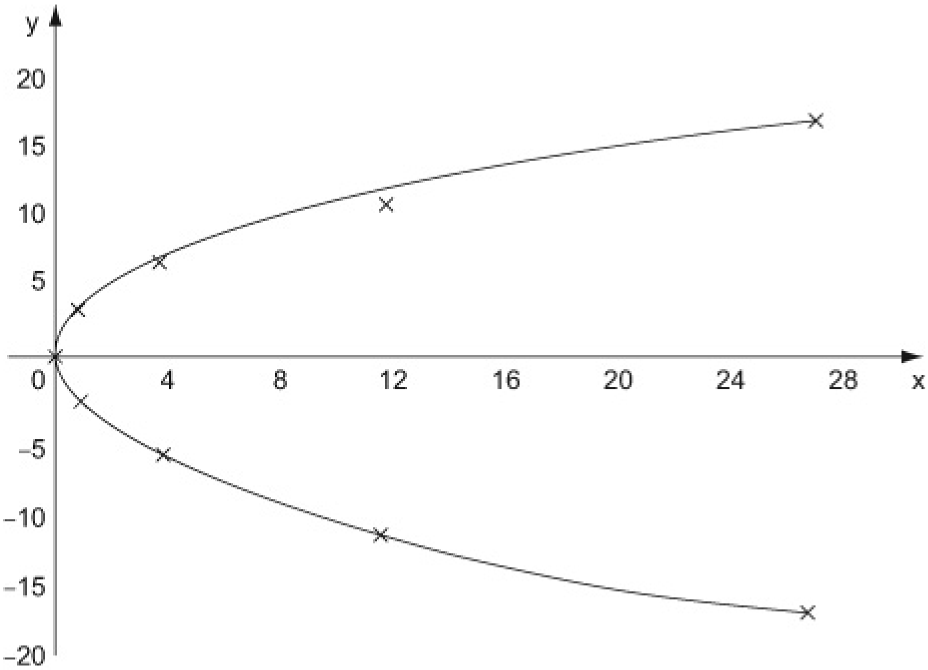
Question:

A curve is given by the parametric equations $x = 3t^2$, $y = 6t$. $t \in \mathbb{R}$. Copy and complete the following table and draw a graph of the curve for $-3 \leq t \leq 3$.

t	−3	−2	−1	−0.5	0	0.5	1	2	3
$x = 3t^2$					0				
$y = 6t$					0				

Solution:

t	−3	−2	−1	−0.5	0	0.5	1	2	3
$x = 3t^2$	27	12	3	0.75	0	0.75	3	12	27
$y = 6t$	−18	−12	−6	−3	0	3	6	12	18



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Quadratic Equations
Exercise A, Question 3

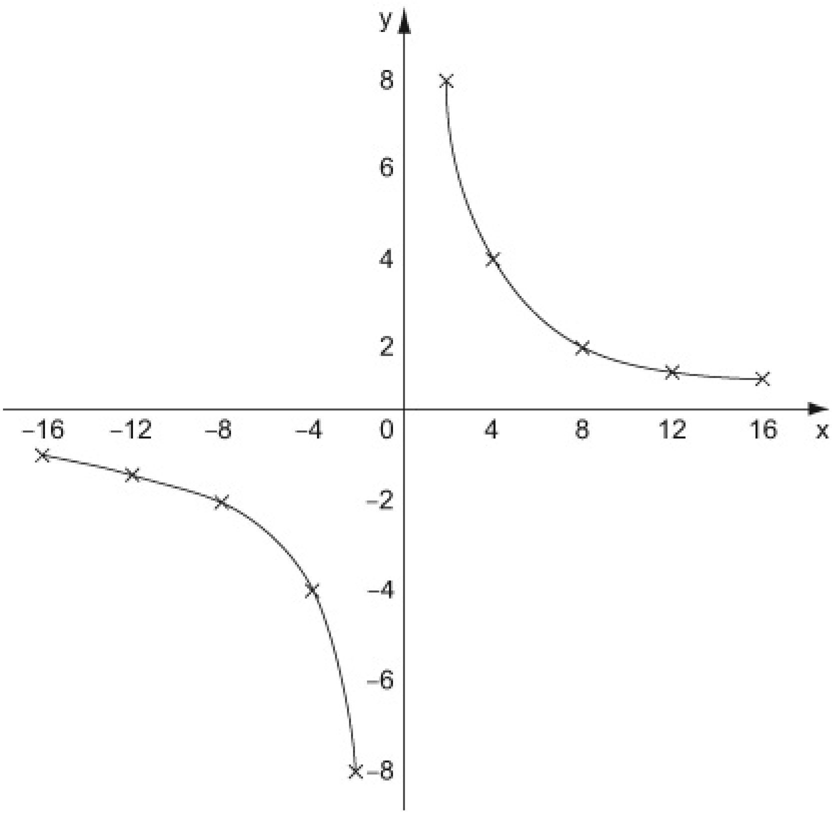
Question:

A curve is given by the parametric equations $x = 4t$, $y = \frac{4}{t}$. $t \in \mathbb{R}$, $t \neq 0$. Copy and complete the following table and draw a graph of the curve for $-4 \leq t \leq 4$.

t	−4	−3	−2	−1	−0.5	0.5	1	2	3	4
$x = 4t$	−16				−2					
$y = \frac{4}{t}$	−1				−8					

Solution:

t	−4	−3	−2	−1	−0.5	0.5	1	2	3	4
$x = 4t$	−16	−12	−8	−4	−2	2	4	8	12	16
$y = \frac{4}{t}$	−1	$-\frac{4}{3}$	−2	−4	−8	8	4	2	$\frac{4}{3}$	1



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Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise A, Question 4

Question:

Find the Cartesian equation of the curves given by these parametric equations.

a $x = 5t^2,\ \ y = 10t$

b $x = \frac{1}{2}t^2,\ \ y = t$

c $x = 50t^2,\ \ y = 100t$

d $x = \frac{1}{5}t^2,\ \ y = \frac{2}{5}t$

e $x = \frac{5}{2}t^2,\ \ y = 5t$

f $x = \sqrt{3}\,t^2,\ \ y = 2\sqrt{3}\,t$

g $x = 4t,\ \ y = 2t^2$

h $x = 6t,\ \ y = 3t^2$

Solution:

a $y = 10t$

So $t = \frac{y}{10}$ **(1)**

$x = 5t^2$ **(2)**

Substitute **(1)** into **(2):**

$$x = 5\left(\frac{y}{10}\right)^2$$

So $x = \frac{5y^2}{100}$ simplifies to $x = \frac{y^2}{20}$

Hence, the Cartesian equation is $y^2 = 20x$.

b $y = t$ **(1)**

$x = \frac{1}{2}t^2$ **(2)**

Substitute **(1)** into **(2):**

$$x = \frac{1}{2}y^2$$

Hence, the Cartesian equation is $y^2 = 2x$.

c $y = 100t$

So $t = \frac{y}{100}$ **(1)**

$x = 50t^2$ **(2)**

Substitute **(1)** into **(2):**

$$x = 50\left(\frac{y}{100}\right)^2$$

So $x = \frac{50y^2}{10000}$ simplifies to $x = \frac{y^2}{200}$

Hence, the Cartesian equation is $y^2 = 200x$.

d $y = \frac{2}{5}t$

So $t = \frac{5y}{2}$ **(1)**

$x = \frac{1}{5}t^2$ **(2)**

Substitute **(1)** into **(2):**

$$x = \frac{1}{5}\left(\frac{5y}{2}\right)^2$$

So $x = \frac{25y^2}{20}$ simplifies to $x = \frac{5y^2}{4}$

Hence, the Cartesian equation is $y^2 = \frac{4}{5}x$.

e $y = 5t$

So $t = \frac{y}{5}$ **(1)**

$x = \frac{5}{2}t^2$ **(2)**

Substitute **(1)** into **(2):**

$$x = \frac{5}{2}\left(\frac{y}{5}\right)^2$$

So $x = \frac{5y^2}{50}$ simplifies to $x = \frac{y^2}{10}$

Hence, the Cartesian equation is $y^2 = 10x$.

f $y = 2\sqrt{3}\,t$

So $t = \frac{y}{2\sqrt{3}}$ **(1)**

$x = \sqrt{3}\,t^2$ **(2)**

Substitute **(1)** into **(2):**

$$x = \sqrt{3}\left(\frac{y}{2\sqrt{3}}\right)^2$$

So $x = \frac{\sqrt{3}y^2}{12}$ gives $y = \frac{12x}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

Hence, the Cartesian equation is $y^2 = 4\sqrt{3}x$.

g $x = 4t$

So $t = \frac{x}{4}$ **(1)**

$y = 2t^2$ **(2)**

Substitute **(1)** into **(2):**

$$y = 2\left(\frac{x}{4}\right)^2$$

So $y = \frac{2x^2}{16}$ simplifies to $y = \frac{x^2}{8}$

Hence, the Cartesian equation is $x^2 = 8y$.

h $x = 6t$

So $t = \frac{x}{6}$ **(1)**

$y = 3t^2$ **(2)**

Substitute **(1)** into **(2):**

$$y = 3\left(\frac{x}{6}\right)^2$$

So $y = \frac{3x^2}{36}$ simplifies to $y = \frac{x^2}{12}$

Hence, the Cartesian equation is $x^2 = 12y$.

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Quadratic Equations
Exercise A, Question 5

Question:

Find the Cartesian equation of the curves given by these parametric equations.

a $x = t,\ \ y = \frac{1}{t},\ \ t \neq 0$

b $x = 7t,\ \ y = \frac{7}{t},\ \ t \neq 0$

c $x = 3\sqrt{5}t,\ \ y = \frac{3\sqrt{5}}{t},\ \ t \neq 0$

d $x = \frac{t}{5},\ \ y = \frac{1}{5t},\ \ t \neq 0$

Solution:

a $xy = t \times \left(\frac{1}{t}\right)$

$xy = \frac{t}{t}$

Hence, the Cartesian equation is $xy = 1$.

b $xy = 7t \times \left(\frac{7}{t}\right)$

$xy = \frac{49t}{t}$

Hence, the Cartesian equation is $xy = 49$.

c $xy = 3\sqrt{5}t \times \left(\frac{3\sqrt{5}}{t}\right)$

$xy = \frac{9(5)t}{t}$

Hence, the Cartesian equation is $xy = 45$.

d $xy = \frac{t}{5} \times \left(\frac{1}{5t}\right)$

$xy = \frac{t}{25t}$

Hence, the Cartesian equation is $xy = \frac{1}{25}$.

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Quadratic Equations
Exercise A, Question 6

Question:

A curve has parametric equations $x = 3t$, $y = \frac{3}{t}$, $t \in \mathbb{R}, t \neq 0$.

a Find the Cartesian equation of the curve.

b Hence sketch this curve.

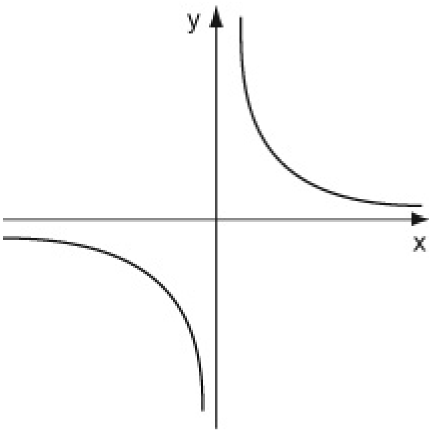
Solution:

a $xy = 3t \times \left(\frac{3}{t}\right)$

$$xy = \frac{9t}{t}$$

Hence, the Cartesian equation is $xy = 9$.

b



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Quadratic Equations
Exercise A, Question 7

Question:

A curve has parametric equations $x = \sqrt{2}t,\ \ y = \frac{\sqrt{2}}{t},\ \ t \in \mathbb{R}, \mathrm{t} \neq 0.$

a Find the Cartesian equation of the curve.

b Hence sketch this curve.

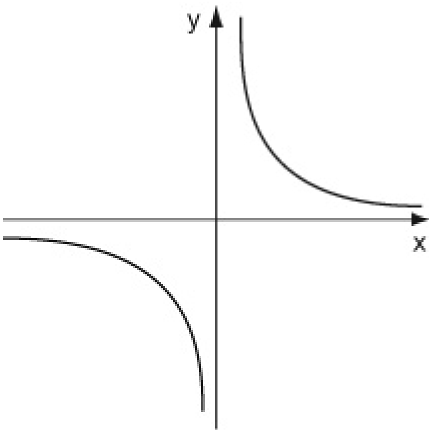
Solution:

a $xy = \sqrt{2}t \times \left(\frac{\sqrt{2}}{t}\right)$

$xy = \frac{2t}{t}$

Hence, the Cartesian equation is $xy = 2$.

b



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Quadratic Equations
Exercise B, Question 1

Question:

Find an equation of the parabola with

a focus (5, 0) and directrix $x+5=0$,

b focus (8, 0) and directrix $x+8=0$,

c focus (1, 0) and directrix $x=-1$,

d focus $\left(\frac{3}{2},\ 0\right)$ and directrix $x=-\frac{3}{2}$,

e focus $\left(\frac{\sqrt{3}}{2},\ 0\right)$ and directrix $x+\frac{\sqrt{3}}{2}=0$.

Solution:

The focus and directrix of a parabola with equation $y^2=4ax$, are $(a,\ 0)$ and $x+a=0$ respectively.

a focus (5, 0) and directrix $x+5=0$.

So $a=5$ and $y^2=4(5)x$.

Hence parabola has equation $y^2=20x$.

b focus (8, 0) and directrix $x+8=0$.

So $a=8$ and $y^2=4(8)x$.

Hence parabola has equation $y^2=32x$.

c focus (1, 0) and directrix $x=-1$ giving $x+1=0$.

So $a=1$ and $y^2=4(1)x$.

Hence parabola has equation $y^2=4x$.

d focus $\left(\frac{3}{2}, 0\right)$ and directrix $x=-\frac{3}{2}$ giving $x+\frac{3}{2}=0$.

So $a=\frac{3}{2}$ and $y^2=4\left(\frac{3}{2}\right)x$.

Hence parabola has equation $y^2=6x$.

e focus $\left(\frac{\sqrt{3}}{2}, 0\right)$ and directrix $x+\frac{\sqrt{3}}{2}=0$.

So $a=\frac{\sqrt{3}}{2}$ and $y^2=4\left(\frac{\sqrt{3}}{2}\right)x$.

Hence parabola has equation $y^2 = 2\sqrt{3}x$.

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Quadratic Equations
Exercise B, Question 2

Question:

Find the coordinates of the focus, and an equation for the directrix of a parabola with these equations.

a $y^2 = 12x$

b $y^2 = 20x$

c $y^2 = 10x$

d $y^2 = 4\sqrt{3}x$

e $y^2 = \sqrt{2}x$

f $y^2 = 5\sqrt{2}x$

Solution:

The focus and directrix of a parabola with equation $y^2 = 4ax$, are $(a, 0)$ and $x + a = 0$ respectively.

a $y^2 = 12x$. So $4a = 12$, gives $a = \frac{12}{4} = 3$.

So the focus has coordinates (3, 0) and the directrix has equation $x + 3 = 0$.

b $y^2 = 20x$. So $4a = 20$, gives $a = \frac{20}{4} = 5$.

So the focus has coordinates (5, 0) and the directrix has equation $x + 5 = 0$.

c $y^2 = 10x$. So $4a = 10$, gives $a = \frac{10}{4} = \frac{5}{2}$.

So the focus has coordinates $\left(\frac{5}{2}, 0\right)$ and the directrix has equation $x + \frac{5}{2} = 0$.

d $y^2 = 4\sqrt{3}x$. So $4a = 4\sqrt{3}$, gives $a = \frac{4\sqrt{3}}{4} = \sqrt{3}$.

So the focus has coordinates $(\sqrt{3}, 0)$ and the directrix has equation $x + \sqrt{3} = 0$.

e $y^2 = \sqrt{2}x$. So $4a = \sqrt{2}$, gives $a = \frac{\sqrt{2}}{4}$.

So the focus has coordinates $\left(\frac{\sqrt{2}}{4}, 0\right)$ and the directrix has equation $x + \frac{\sqrt{2}}{4} = 0$.

f $y^2 = 5\sqrt{2}x$. So $4a = 5\sqrt{2}$, gives $a = \frac{5\sqrt{2}}{4}$.

So the focus has coordinates $\left(\frac{5\sqrt{2}}{4}, 0\right)$ and the directrix has equation $x + \frac{5\sqrt{2}}{4} = 0$.

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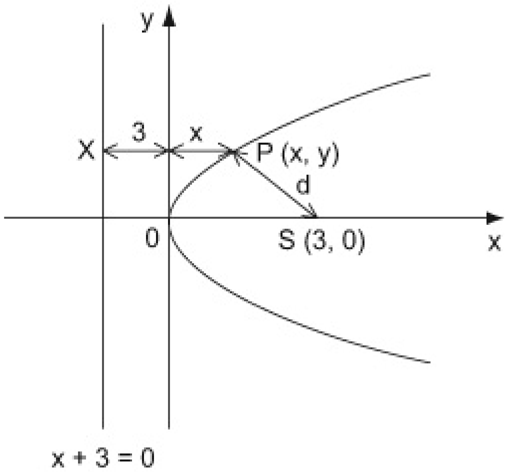
Quadratic Equations
Exercise B, Question 3

Question:

A point $P(x, y)$ obeys a rule such that the distance of P to the point (3, 0) is the same as the distance of P to the straight line $x + 3 = 0$. Prove that the locus of P has an equation of the form $y^2 = 4ax$, stating the value of the constant a.

Solution:



The (shortest) distance of P to the line $x + 3 = 0$ is the distance XP.

The distance SP is the same as the distance XP.

The line XP is horizontal and has distance $XP = x + 3$.

The locus of P is the curve shown.

From sketch the locus satisfies $SP = XP$.

This means the distance SP is the same as the distance XP.

Therefore, $SP^2 = XP^2$.

So, $(x - 3)^2 + (y - 0)^2 = (x - -3)^2$.

Use $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ on $SP^2 = XP^2$, where $S(3, 0)$, $P(x, y)$, and $X(-3, y)$.

$$x^2 - 6x + 9 + y^2 = x^2 + 6x + 9$$
$$-6x + y^2 = 6x$$

which simplifies to $y^2 = 12x$.

This is in the form $y^2 = 4ax$.

So $4a = 12$, gives $a = \frac{12}{4} = 3$.

So, the locus of P has an equation of the form $y^2 = 4ax$, where $a = 3$.

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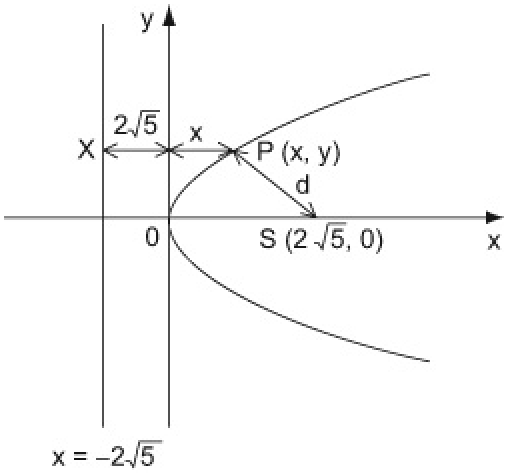
Quadratic Equations
Exercise B, Question 4

Question:

A point $P(x, y)$ obeys a rule such that the distance of P to the point $(2\sqrt{5},\ 0)$ is the same as the distance of P to the straight line $x = -2\sqrt{5}$. Prove that the locus of P has an equation of the form $y^2 = 4ax$, stating the value of the constant a.

Solution:



The (shortest) distance of P to the line $x = -2\sqrt{5}$ or $x + 2\sqrt{5} = 0$ is the distance XP.

The distance SP is the same as the distance XP.

The line XP is horizontal and has distance $XP = x + 2\sqrt{5}$.

The locus of P is the curve shown.

From sketch the locus satisfies $SP = XP$.

This means the distance SP is the same as the distance XP.

Therefore, $SP^2 = XP^2$.

So, $(x - 2\sqrt{5})^2 + (y - 0)^2 = (x - -2\sqrt{5})^2$.

Use $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ on $SP^2 = XP^2$, where $S(2\sqrt{5}, 0)$, $P(x, y)$, and $X(-2\sqrt{5}, y)$.

$$x^2 - 4\sqrt{5}x + 20 + y^2 = x^2 + 4\sqrt{5}x + 20$$
$$-4\sqrt{5}x + y^2 = 4\sqrt{5}x$$

which simplifies to $y^2 = 8\sqrt{5}x$.

This is in the form $y^2 = 4ax$.

So $4a = 8\sqrt{5}$, gives $a = \frac{8\sqrt{5}}{4} = 2\sqrt{5}$.

So, the locus of P has an equation of the form $y^2 = 4ax$, where $a = 2\sqrt{5}$.

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Quadratic Equations
Exercise B, Question 5

Question:

A point $P(x, y)$ obeys a rule such that the distance of P to the point (0, 2) is the same as the distance of P to the straight line $y = -2$.

a Prove that the locus of P has an equation of the form $y = kx^2$, stating the value of the constant k.

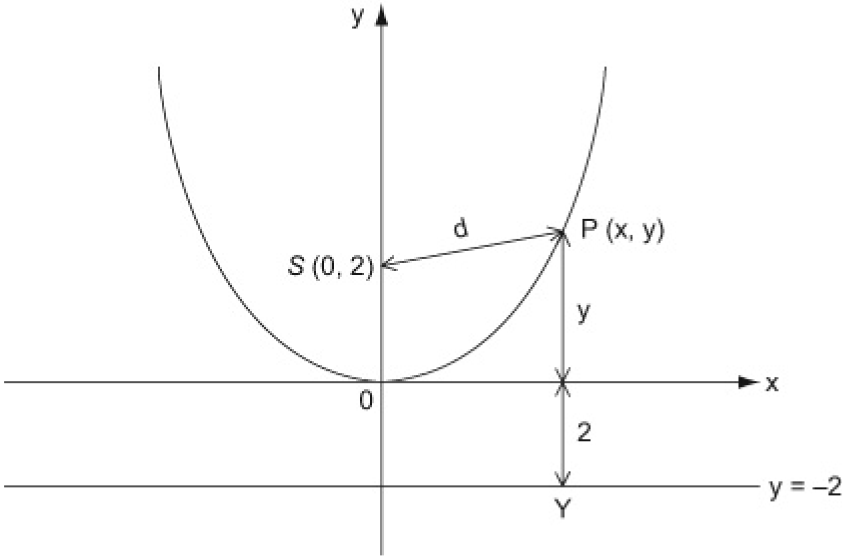
Given that the locus of P is a parabola,

b state the coordinates of the focus of P, and an equation of the directrix to P,

c sketch the locus of P with its focus and its directrix.

Solution:

a



The (shortest) distance of P to the line $y = -2$ is the distance YP.

The distance SP is the same as the distance YP.

The line YP is vertical and has distance $YP = y + 2$.

The locus of P is the curve shown.

From sketch the locus satisfies $SP = YP$.

This means the distance SP is the same as the distance YP.

Therefore, $SP^2 = YP^2$.

So, $(x - 0)^2 + (y - 2)^2 = (y - -2)^2$.

Use $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ on $SP^2 = YP^2$, where $S(0, 2)$, $P(x, y)$, and $Y(x, -2)$.

$$x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$$
$$x^2 - 4y = 4y$$

which simplifies to $x^2 = 8y$ and then $y = \frac{1}{8}x^2$.

So, the locus of P has an equation of the form $y = \frac{1}{8}x^2$, where

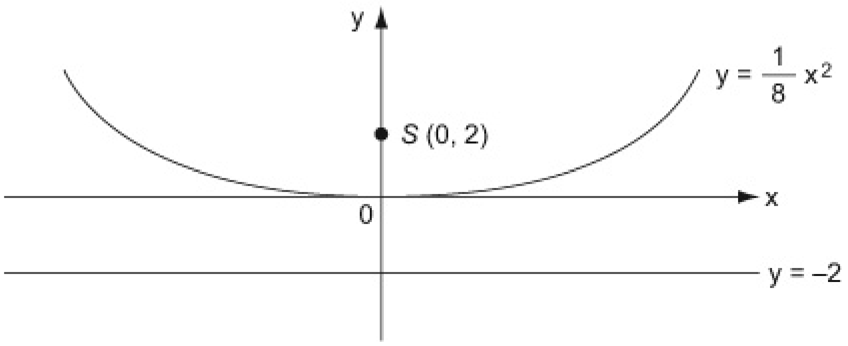
$k = \frac{1}{8}$.

b The focus and directrix of a parabola with equation $y^2 = 4ax$, are $(a, 0)$ and $x + a = 0$ respectively. Therefore it follows that the focus and directrix of a parabola with equation $x^2 = 4ay$, are $(0, a)$ and $y + a = 0$ respectively.

So the focus has coordinates (0, 2) and the directrix has equation $y + 2 = 0$.

$x^2 = 8y$ is in the form $x^2 = 4ay$.
So $4a = 8$, gives $a = \frac{8}{4} = 2$.

c



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Quadratic Equations
Exercise C, Question 1

Question:

The line $y = 2x - 3$ meets the parabola $y^2 = 3x$ at the points P and Q.

Find the coordinates of P and Q.

Solution:

Line: $y = 2x - 3$ **(1)**

Curve: $y^2 = 3x$ **(2)**

Substituting **(1)** into **(2)** gives

$$(2x - 3)^2 = 3x$$
$$(2x - 3)(2x - 3) = 3x$$
$$4x^2 - 12x + 9 = 3x$$
$$4x^2 - 15x + 9 = 0$$
$$(x - 3)(4x - 3) = 0$$
$$x = 3, \frac{3}{4}$$

When $x = 3$, $y = 2(3) - 3 = 3$

When $x = \frac{3}{4}$, $y = 2\left(\frac{3}{4}\right) - 3 = -\frac{3}{2}$

Hence the coordinates of P and Q are (3, 3) and $\left(\frac{3}{4}, -\frac{3}{2}\right)$.

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Quadratic Equations
Exercise C, Question 2

Question:

The line $y = x + 6$ meets the parabola $y^2 = 32x$ at the points *A* and *B*. Find the exact length *AB* giving your answer as a surd in its simplest form.

Solution:

Line: $y = x + 6$ **(1)**

Curve: $y^2 = 32x$ **(2)**

Substituting **(1)** into **(2)** gives

$$(x+6)^2 = 32x$$
$$(x+6)(x+6) = 32x$$
$$x^2 + 12x + 36 = 32x$$
$$x^2 - 20x + 36 = 0$$
$$(x-2)(x-18) = 0$$
$$x = 2, 18$$

When $x = 2$, $y = 2 + 6 = 8$.

When $x = 18$, $y = 18 + 6 = 24$.

Hence the coordinates of *A* and *B* are (2, 8) and (18, 24).

$$\begin{aligned} AB &= \sqrt{(18-2)^2 + (24-8)^2} \text{ Use d} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \\ &= \sqrt{16^2 + 16^2} \\ &= \sqrt{2(16)^2} \\ &= 16\sqrt{2} \end{aligned}$$

Hence the exact length *AB* is $16\sqrt{2}$.

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Quadratic Equations
Exercise C, Question 3

Question:

The line $y = x - 20$ meets the parabola $y^2 = 10x$ at the points A and B. Find the coordinates of A and B. The mid-point of AB is the point M. Find the coordinates of M.

Solution:

Line: $y = x - 20$ **(1)**

Curve: $y^2 = 10x$ **(2)**

Substituting **(1)** into **(2)** gives

$(x-20)^2 = 10x$
$(x-20)(x-20) = 10x$
$x^2 - 40x + 400 = 10x$
$x^2 - 50x + 400 = 0$
$(x-10)(x-40) = 0$
$x = 10, 40$

When $x = 10$, $y = 10 - 20 = -10$.

When $x = 40$, $y = 40 - 20 = 20$.

Hence the coordinates of A and B are (10, −10) and (40, 20).

The midpoint of A and B is $\left(\frac{10+40}{2}, \frac{-10+20}{2}\right) = (25, 5)$. Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Hence the coordinates of M are (25, 5).

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Quadratic Equations
Exercise C, Question 4

Question:

The parabola C has parametric equations $x = 6t^2$, $y = 12t$. The focus to C is at the point S.

a Find a Cartesian equation of C.

b State the coordinates of S and the equation of the directrix to C.

c Sketch the graph of C.

The points P and Q are both at a distance 9 units away from the directrix of the parabola.

d State the distance PS.

e Find the exact length PQ, giving your answer as a surd in its simplest form.

f Find the area of the triangle PQS, giving your answer in the form $k\sqrt{2}$, where k is an integer.

Solution:

a $y = 12t$

So $t = \frac{y}{12}$ **(1)**

$x = 6t^2$ **(2)**

Substitute **(1)** into **(2)**:

$$x = 6\left(\frac{y}{12}\right)^2$$

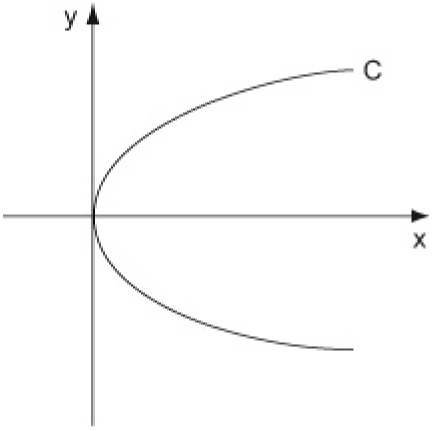
So $x = \frac{6y^2}{144}$ simplifies to $x = \frac{y^2}{24}$

Hence, the Cartesian equation is $y^2 = 24x$.

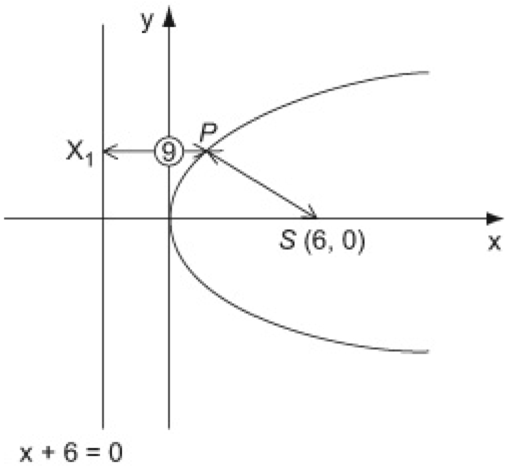
b $y^2 = 24x$. So $4a = 24$, gives $a = \frac{24}{4} = 6$.

So the focus S, has coordinates (6, 0) and the directrix has equation $x + 6 = 0$.

c



d



The (shortest) distance of P to the line $x + 6 = 0$ is the distance X_1P.

Therefore $X_1P = 9$.

The distance PS is the same as the distance X_1P, by the focus-directrix property.

Hence the distance $PS = 9$.

e Using diagram in (d), the x-coordinate of P and Q is $x = 9 - 6 = 3$.

When $x = 3$, $y^2 = 24(3) = 72$.

$$\begin{aligned}\text{Hence } y &= \pm\sqrt{72}\\ &= \pm\sqrt{36}\sqrt{2}\\ &= \pm 6\sqrt{2}\end{aligned}$$

So the coordinates are of P and Q are $(3, 6\sqrt{2})$ and $(3, -6\sqrt{2})$.

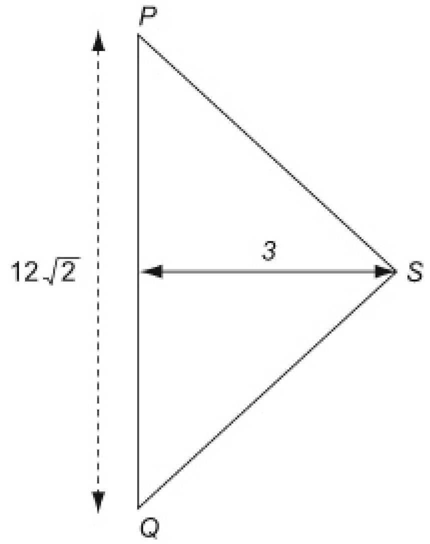
As P and Q are vertically above each other then

$$\begin{aligned}PQ &= 6\sqrt{2} - -6\sqrt{2}\\ &= 12\sqrt{2}.\end{aligned}$$

Hence, the distance PQ is $12\sqrt{2}$.

f Drawing a diagram of the triangle PQS gives:

The x-coordinate of P and Q is 3 and the x-coordinate of S is 6.



Hence the height of the triangle is height $= 6 - 3 = 3$.

The length of the base is $12\sqrt{2}$.

$$\begin{aligned}\text{Area} &= \frac{1}{2}(12\sqrt{2})(3)\\ &= \frac{1}{2}(36\sqrt{2})\\ &= 18\sqrt{2}.\end{aligned}$$

Therefore the area of the triangle is $18\sqrt{2}$, where $k = 18$.

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Quadratic Equations
Exercise C, Question 5

Question:

The parabola C has equation $y^2 = 4ax$, where a is a constant. The point $\left(\frac{5}{4}t^2,\ \frac{5}{2}t\right)$ is a general point on C.

a Find a Cartesian equation of C.

The point P lies on C with y-coordinate 5.

b Find the x-coordinate of P.

The point Q lies on the directrix of C where $y = 3$. The line l passes through the points P and Q.

c Find the coordinates of Q.

d Find an equation for l, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

Solution:

a $P\left(\frac{5}{4}t^2, \frac{5}{2}t\right)$. Substituting $x = \frac{5}{4}t^2$ and $y = \frac{5}{2}t$ into $y^2 = 4ax$ gives,

$$\left(\frac{5}{2}t\right)^2 = 4a\left(\frac{5}{4}t^2\right) \Rightarrow \frac{25t^2}{4} = 5at^2 \Rightarrow \frac{25}{4} = 5a \Rightarrow \frac{5}{4} = a$$

When $a = \frac{5}{4}, y^2 = 4\left(\frac{5}{4}\right)x \Rightarrow y^2 = 5x$

The Cartesian equation of C is $y^2 = 5x$.

b When $y = 5, (5)^2 = 5x \Rightarrow \frac{25}{5} = x \Rightarrow x = 5.$

The x-coordinate of P is 5.

c As $a = \frac{5}{4}$, the equation of the directrix of C is $x + \frac{5}{4} = 0$ or $x = -\frac{5}{4}$.

Therefore the coordinates of Q are $\left(-\frac{5}{4}, 3\right)$.

d The coordinates of P and Q are (5, 5) and $\left(-\frac{5}{4}, 3\right)$.

$$m_l = m_{PQ} = \frac{3-5}{-\frac{5}{4}-5} = \frac{-2}{-\frac{25}{4}} = \frac{8}{25}$$

$$\begin{aligned}
l &: y - 5 = \frac{8}{25}(x-5) \\
l &: 25y - 125 = 8(x-5) \\
l &: 25y - 125 = 8x - 40 \\
l &: 0 = 8x - 25y - 40 + 125 \\
l &: 0 = 8x - 25y + 85
\end{aligned}$$

An equation for l is $8x - 25y + 85 = 0$.

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Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise C, Question 6

Question:

A parabola C has equation $y^2 = 4x$. The point S is the focus to C.

a Find the coordinates of S.

The point P with y-coordinate 4 lies on C.

b Find the x-coordinate of P.

The line l passes through S and P.

c Find an equation for l, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

The line l meets C again at the point Q.

d Find the coordinates of Q.

e Find the distance of the directrix of C to the point Q.

Solution:

a $y^2 = 4x$. So $4a = 4$, gives $a = \frac{4}{4} = 1$.

So the focus S, has coordinates (1, 0).

Also note that the directrix has equation $x + 1 = 0$.

b Substituting $y = 4$ into $y^2 = 4x$ gives:

$16 = 4x \Rightarrow x = \frac{16}{4} = 4.$

The x-coordinate of P is 4.

c The line l goes through $S(1, 0)$ and $P(4, 4)$.

Hence gradient of l, $m_l = \frac{4-0}{4-1} = \frac{4}{3}$

$$\begin{aligned}
\text{Hence, } y - 0 &= \tfrac{4}{3}(x-1) \\
3y &= 4(x-1) \\
3y &= 4x - 4 \\
0 &= 4x - 3y - 4
\end{aligned}$$

The line l has equation $4x - 3y - 4 = 0$.

d Line $l: 4x - 3y - 4 = 0$ **(1)**

Curve : $y^2 = 4x$ **(2)**

Substituting **(2)** into **(1)** gives

$y^2 - 3y - 4 = 0$
$(y-4)(y+1) = 0$
$y = 4, -1$

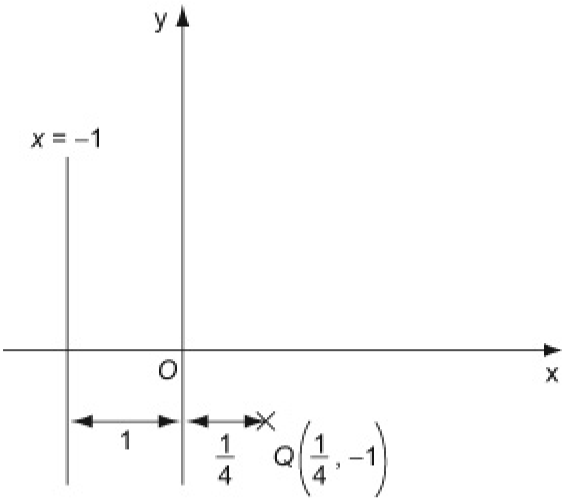
At P, it is already known that $y = 4$. So at Q, $y = -1$.

Substituting $y = -1$ into $y^2 = 4x$ gives

$(-1)^2 = 4x \Rightarrow x = \frac{1}{4}$.

Hence the coordinates of Q are $\left(\frac{1}{4}, -1\right)$.

e The directrix of C has equation $x + 1 = 0$ or $x = -1$. Q has coordinates $\left(\frac{1}{4}, -1\right)$.



From the diagram, distance $= 1 + \frac{1}{4} = \frac{5}{4}$.

Therefore the distance of the directrix of C to the point Q is $\frac{5}{4}$.

Solutionbank FP1

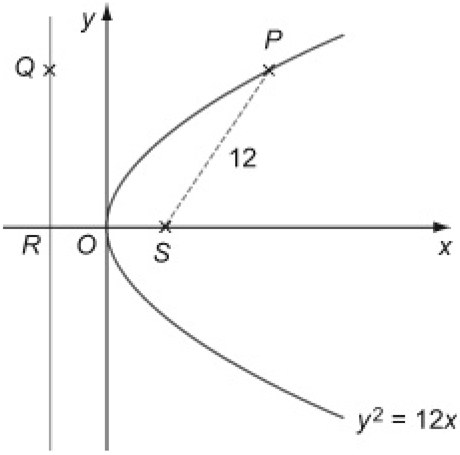
Edexcel AS and A Level Modular Mathematics

Quadratic Equations

Exercise C, Question 7

Question:

The diagram shows the point P which lies on the parabola C with equation $y^2 = 12x$.



The point S is the focus of C. The points Q and R lie on the directrix to C. The line segment QP is parallel to the line segment RS as shown in the diagram. The distance of PS is 12 units.

a Find the coordinates of R and S.

b Hence find the exact coordinates of P and Q.

c Find the area of the quadrilateral $PQRS$, giving your answer in the form $k\sqrt{3}$, where k is an integer.

Solution:

a $y^2 = 12x$. So $4a = 12$, gives $a = \frac{12}{4} = 3$.

Therefore the focus S has coordinates (3, 0) and an equation of the directrix of C is $x + 3 = 0$ or $x = -3$. The coordinates of R are $(-3, 0)$ as R lies on the x-axis.

b The directrix has equation $x = -3$. The (shortest) distance of P to the directrix is the distance PQ. The distance $SP = 12$. The focus-directrix property implies that $SP = PQ = 12$.

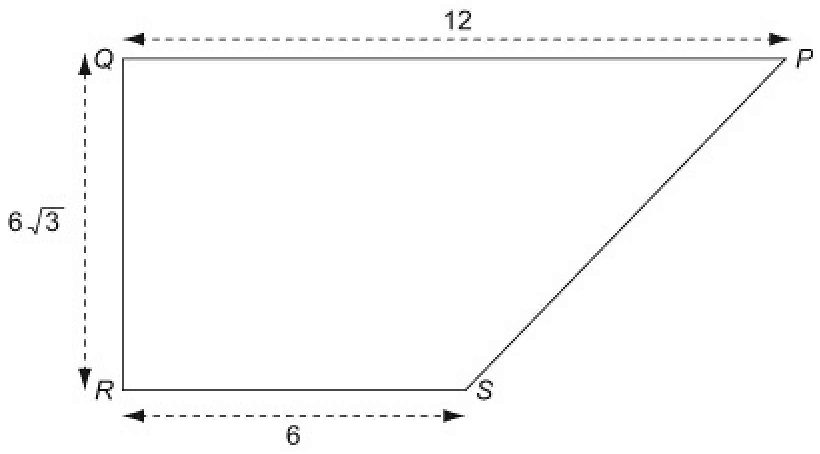
Therefore the x-coordinate of P is $x = 12 - 3 = 9$.

As P lies on C, when $x = 9, y^2 = 12(9) \Rightarrow y^2 = 108$

As $y > 0$, $y = \sqrt{108} = \sqrt{36}\sqrt{3} = 6\sqrt{3} \Rightarrow P(9, 6\sqrt{3})$

Hence the exact coordinates of P are $(9, 6\sqrt{3})$ and the coordinates of Q are $(-3, 6\sqrt{3})$.

c



$$
\begin{aligned}
\text{Area}(PQRS) &= \frac{1}{2}(6+12)6\sqrt{3} \\
&= \frac{1}{2}(18)(6\sqrt{3}) \\
&= (9)(6\sqrt{3}) \\
&= 54\sqrt{3}
\end{aligned}
$$

The area of the quadrilateral $PQRS$ is $54\sqrt{3}$ and $k = 54$.

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Quadratic Equations
Exercise C, Question 8

Question:

The points $P(16, 8)$ and $Q(4, b)$, where $b < 0$ lie on the parabola C with equation $y^2 = 4ax$.

a Find the values of a and b.

P and Q also lie on the line l. The mid-point of PQ is the point R.

b Find an equation of l, giving your answer in the form $y = mx + c$, where m and c are constants to be determined.

c Find the coordinates of R.

The line n is perpendicular to l and passes through R.

d Find an equation of n, giving your answer in the form $y = mx + c$, where m and c are constants to be determined.

The line n meets the parabola C at two points.

e Show that the x-coordinates of these two points can be written in the form $x = \lambda \pm \mu\sqrt{13}$, where λ and μ are integers to be determined.

Solution:

a $P(16, 8)$. Substituting $x = 16$ and $y = 8$ into $y^2 = 4ax$ gives,

$(8)^2 = 4a(16) \Rightarrow 64 = 64a \Rightarrow a = \frac{64}{64} = 1.$

$Q(4, b)$. Substituting $x = 4$, $y = b$ and $a = 1$ into $y^2 = 4ax$ gives,

$b^2 = 4(1)(4) = 16 \Rightarrow b = \pm\sqrt{16} \Rightarrow b = \pm 4$. As $b < 0$, $b = -4$.

Hence, $a = 1$, $b = -4$.

b The coordinates of P and Q are $(16, 8)$ and $(4, -4)$.

$m_l = m_{PQ} = \frac{-4-8}{4-16} = \frac{-12}{-12} = 1$

$l : y - 8 = 1(x - 16)$

$l : y = x - 8$

l has equation $y = x - 8$.

c R has coordinates $\left(\frac{16+4}{2}, \frac{8+-4}{2}\right) = (10, 2)$.

d As n is perpendicular to l, $m_n = -1$

$n : y - 2 = -1(x - 10)$

$n : y - 2 = -x + 10$

$n : y = -x + 12$

n has equation $y = -x + 12$.

e Line n : $y = -x + 12$ **(1)**

Parabola C : $y^2 = 4x$ **(2)**

Substituting **(1)** into **(2)** gives

$(-x + 12)^2 = 4x$
$x^2 - 12x - 12x + 144 = 4x$
$x^2 - 28x + 144 = 0$
$(x - 14)^2 - 196 + 144 = 0$
$(x - 14)^2 - 52 = 0$
$(x - 14)^2 = 52$
$x - 14 = \pm\sqrt{52}$
$x - 14 = \pm\sqrt{4}\sqrt{13}$
$x - 14 = \pm 2\sqrt{13}$
$x = 14 \pm 2\sqrt{13}$

The x coordinates are $x = 14 \pm 2\sqrt{13}$.

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Quadratic Equations
Exercise D, Question 1

Question:

Find the equation of the tangent to the curve

a $y^2 = 4x$ at the point (16, 8)

b $y^2 = 8x$ at the point $(4,\ 4\sqrt{2})$

c $xy = 25$ at the point (5, 5)

d $xy = 4$ at the point where $x = \frac{1}{2}$

e $y^2 = 7x$ at the point $(7,\ -7)$

f $xy = 16$ at the point where $x = 2\sqrt{2}$.

Give your answers in the form $ax + by + c = 0$.

Solution:

a As $y > 0$ in the coordinates (16, 8), then

$y^2 = 4x \Rightarrow y = \sqrt{4x} = \sqrt{4}\sqrt{x} = 2x^{\frac{1}{2}}$

So $y = 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$

At (16, 8), $m_T = \frac{dy}{dx} = \frac{1}{\sqrt{16}} = \frac{1}{4}$.

T: $y - 8 = \frac{1}{4}(x - 16)$

T: $4y - 32 = x - 16$

T: $0 = x - 4y - 16 + 32$

T: $x - 4y + 16 = 0$

Therefore, the equation of the tangent is $x - 4y + 16 = 0$.

b As $y > 0$ in the coordinates $(4, 4\sqrt{2})$, then

$y^2 = 8x \Rightarrow y = \sqrt{8x} = \sqrt{8}\sqrt{x} = \sqrt{4}\sqrt{2}\sqrt{x} = 2\sqrt{2}x^{\frac{1}{2}}$

So $y = 2\sqrt{2}x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\sqrt{2}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{2}x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{\sqrt{2}}{\sqrt{x}}$

At $(4, 4\sqrt{2})$, $m_T = \frac{dy}{dx} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}$.

T: $y - 4\sqrt{2} = \frac{\sqrt{2}}{2}(x - 4)$

T: $2y - 8\sqrt{2} = \sqrt{2}(x - 4)$

T: $2y - 8\sqrt{2} = \sqrt{2}x - 4\sqrt{2}$

T: $0 = \sqrt{2}x - 2y - 4\sqrt{2} + 8\sqrt{2}$

T: $\sqrt{2}x - 2y + 4\sqrt{2} = 0$

Therefore, the equation of the tangent is $\sqrt{2}x - 2y + 4\sqrt{2} = 0$.

c $xy = 25 \Rightarrow y = 25x^{-1}$

$\frac{dy}{dx} = -25x^{-2} = -\frac{25}{x^2}$

At (5, 5), $m_T = \frac{dy}{dx} = -\frac{25}{5^2} = -\frac{25}{25} = -1$

T: $y - 5 = -1(x - 5)$

T: $y - 5 = -x + 5$

T: $x + y - 5 - 5 = 0$

T: $x + y - 10 = 0$

Therefore, the equation of the tangent is $x + y - 10 = 0$.

d $xy = 4 \Rightarrow y = 4x^{-1}$

$\frac{dy}{dx} = -4x^{-2} = -\frac{4}{x^2}$

At $x = \frac{1}{2}$, $m_T = \frac{dy}{dx} = -\frac{4}{\left(\frac{1}{2}\right)^2} = -\frac{4}{\left(\frac{1}{4}\right)} = -16$

When $x = \frac{1}{2}$, $y = \frac{4}{\left(\frac{1}{2}\right)} = 8 \Rightarrow \left(\frac{1}{2}, 8\right)$

T: $y - 8 = -16\left(x - \frac{1}{2}\right)$

T: $y - 8 = -16x + 8$

T: $16x + y - 8 - 8 = 0$

T: $16x + y - 16 = 0$

Therefore, the equation of the tangent is $16x + y - 16 = 0$.

e As $y < 0$ in the coordinates $(7, -7)$, then

$y^2 = 7x \Rightarrow y = -\sqrt{7x} = -\sqrt{7}\,\sqrt{x} = -\sqrt{7}\,x^{\frac{1}{2}}$

So $y = -\sqrt{7}\,x^{\frac{1}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{7}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = -\frac{\sqrt{7}}{2}x^{-\frac{1}{2}}$

So, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sqrt{7}}{2\sqrt{x}}$

At $(7, -7)$, $m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sqrt{7}}{2\sqrt{7}} = -\frac{1}{2}$.

T: $y + 7 = -\frac{1}{2}(x - 7)$

T: $2y + 14 = -1(x - 7)$

T: $2y + 14 = -x + 7$

T: $x + 2y + 14 - 7 = 0$

T: $x + 2y + 7 = 0$

Therefore, the equation of the tangent is $x + 2y + 7 = 0$.

f $xy = 16 \Rightarrow y = 16x^{-1}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -16x^{-2} = -\frac{16}{x^2}$

At $x = 2\sqrt{2}$, $m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{16}{(2\sqrt{2})^2} = -\frac{16}{8} = -2$

When $x = 2\sqrt{2}$, $y = \frac{16}{2\sqrt{2}} = \frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{\sqrt{2}\,\sqrt{2}} = 4\sqrt{2} \Rightarrow (2\sqrt{2}, 4\sqrt{2})$

T: $y - 4\sqrt{2} = -2(x - 2\sqrt{2})$

T: $y - 4\sqrt{2} = -2x + 4\sqrt{2}$

T: $2x + y - 4\sqrt{2} - 4\sqrt{2} = 0$

T: $2x + y - 8\sqrt{2} = 0$

Therefore, the equation of the tangent is $2x + y - 8\sqrt{2} = 0$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise D, Question 2

Question:

Find the equation of the normal to the curve

a $y^2 = 20x$ at the point where $y = 10$,

b $xy = 9$ at the point $\left(-\frac{3}{2},\ -6\right)$.

Give your answers in the form $ax + by + c = 0$, where a, b and c are integers.

Solution:

a Substituting $y = 10$ into $y^2 = 20x$ gives

$(10)^2 = 20x \Rightarrow x = \frac{100}{20} = 5 \Rightarrow (5, 10)$

As $y > 0$, then

$y^2 = 20x \Rightarrow y = \sqrt{20x} = \sqrt{20}\sqrt{x} = \sqrt{4}\sqrt{5}\sqrt{x} = 2\sqrt{5}x^{\frac{1}{2}}$

So $y = 2\sqrt{5}x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\sqrt{5}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{5}x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{\sqrt{5}}{\sqrt{x}}$

At $(5, 10)$, $m_T = \frac{dy}{dx} = \frac{\sqrt{5}}{\sqrt{5}} = 1.$

Gradient of tangent at $(5, 10)$ is $m_T = 1.$

So gradient of normal is $m_N = -1.$

N: $y - 10 = -1(x - 5)$

N: $y - 10 = -x + 5$

N: $x + y - 10 - 5 = 0$

N: $x + y - 15 = 0$

Therefore, the equation of the normal is $x + y - 15 = 0.$

b $xy = 9 \Rightarrow y = 9x^{-1}$

$\frac{dy}{dx} = -9x^{-2} = -\frac{9}{x^2}$

At $x=-\frac{3}{2}$, $m_T=\frac{dy}{dx}=-\frac{9}{(-\frac{3}{2})^2}=-\frac{9}{(\frac{9}{4})}=-\frac{36}{9}=-4$

Gradient of tangent at $(-\frac{3}{2},-6)$ is $m_T=-4$.

So gradient of normal is $m_N=\frac{-1}{-4}=\frac{1}{4}$.

N: $y+6=\frac{1}{4}(x+\frac{3}{2})$

N: $4y+24=x+\frac{3}{2}$

N: $8y+48=2x+3$

N: $0=2x-8y+3-48$

N: $0=2x-8y-45$

Therefore, the equation of the normal is $2x-8y-45=0$.

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Quadratic Equations
Exercise D, Question 3

Question:

The point $P(4, 8)$ lies on the parabola with equation $y^2 = 4ax$. Find

a the value of a,

b an equation of the normal to C at P.

The normal to C at P cuts the parabola again at the point Q. Find

c the coordinates of Q,

d the length PQ, giving your answer as a simplified surd.

Solution:

a Substituting $x = 4$ and $y = 8$ into $y^2 = 4ax$ gives

$$(8)^2 = 4(a)(4) \Rightarrow 64 = 16a \Rightarrow a = \frac{64}{16} = 4$$

So, $a = 4$.

b When $a = 4$, $y^2 = 4(4)x \Rightarrow y^2 = 16x$.

For $P(4, 8), y > 0$, so

$$y^2 = 16x \Rightarrow y = \sqrt{16x} = \sqrt{16}\sqrt{x} = 4\sqrt{x} = 4x^{\frac{1}{2}}$$

So $y = 4x^{\frac{1}{2}}$

$$\frac{dy}{dx} = 4\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = 2x^{-\frac{1}{2}}$$

So, $\frac{dy}{dx} = \frac{2}{\sqrt{x}}$

At $P(4, 8), m_T = \frac{dy}{dx} = \frac{2}{\sqrt{4}} = \frac{2}{2} = 1.$

Gradient of tangent at $P(4, 8)$ is $m_T = 1$.

So gradient of normal at $P(4, 8)$ is $m_N = -1$.

N: $y - 8 = -1(x - 4)$

N: $y - 8 = -x + 4$

N: $y = -x + 4 + 8$

N: $y = -x + 12$

Therefore, the equation of the normal to C at P is $y = -x + 12$.

c Normal **N:** $y = -x + 12$ **(1)**

Parabola: $y^2 = 16x$ **(2)**

Multiplying **(1)** by 16 gives

$16y = -16x + 192$

Substituting **(2)** into this equation gives

$$16y = -y^2 + 192$$
$$y^2 + 16y - 192 = 0$$
$$(y + 24)(y - 8) = 0$$
$$y = -24, 8$$

At P, it is already known that $y = 8$. So at Q, $y = -24$.

Substituting $y = -24$ into $y^2 = 16x$ gives

$$(-24)^2 = 16x \Rightarrow 576 = 16x \Rightarrow x = \frac{576}{16} = 36.$$

Hence the coordinates of Q are $(36, -24)$.

d The coordinates of P and Q are (4, 8) and $(36, -24)$.

$$\begin{aligned} AB &= \sqrt{(36-4)^2 + (-24-8)^2} \quad \text{Use } d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \\ &= \sqrt{32^2 + (-32)^2} \\ &= \sqrt{2(32)^2} \\ &= \sqrt{2}\sqrt{(32)^2} \\ &= 32\sqrt{2} \end{aligned}$$

Hence the exact length AB is $32\sqrt{2}$.

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Quadratic Equations
Exercise D, Question 4

Question:

The point $A(-2, \ -16)$ lies on the rectangular hyperbola H with equation $xy = 32$.

a Find an equation of the normal to *H* at *A*.

The normal to *H* at *A* meets *H* again at the point *B*.

b Find the coordinates of *B*.

Solution:

a $xy = 32 \Rightarrow y = 32x^{-1}$

$\frac{dy}{dx} = -32x^{-2} = -\frac{32}{x^2}$

At $A(-2, -16)$, $m_T = \frac{dy}{dx} = -\frac{32}{2^2} = -\frac{32}{4} = -8$

Gradient of tangent at $A(-2, -16)$ is $m_T = -8$.

So gradient of normal at $A(-2, -16)$ is $m_N = \frac{-1}{-8} = \frac{1}{8}$.

N: $y + 16 = \frac{1}{8}(x + 2)$

N: $8y + 128 = x + 2$

N: $0 = x - 8y + 2 - 128$

N: $0 = x - 8y - 126$

The equation of the normal to *H* at *A* is $x - 8y - 126 = 0$.

b Normal **N:** $x - 8y - 126 = 0$ **(1)**

Hyperbola H : $xy = 32$ **(2)**

Rearranging **(2)** gives

$y = \frac{32}{x}$

Substituting this equation into **(1)** gives

$$x - 8\left(\frac{32}{x}\right) - 126 = 0$$
$$x - \left(\frac{256}{x}\right) - 126 = 0$$

Multiplying both sides by x gives

$x^2 - 256 - 126x = 0$
$x^2 - 126x - 256 = 0$
$(x - 128)(x + 2) = 0$
$x = 128, -2$

At A, it is already known that $x = -2$. So at B, $x = 128$.

Substituting $x = 128$ into $y = \frac{32}{x}$ gives

$y = \frac{32}{128} = \frac{1}{4}$.

Hence the coordinates of B are $\left(128, \frac{1}{4}\right)$.

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Quadratic Equations
Exercise D, Question 5

Question:

The points $P(4, 12)$ and $Q(-8, -6)$ lie on the rectangular hyperbola H with equation $xy = 48$.

a Show that an equation of the line PQ is $3x - 2y + 12 = 0$.

The point A lies on H. The normal to H at A is parallel to the chord PQ.

b Find the exact coordinates of the two possible positions of A.

Solution:

a The points P and Q have coordinates $P(4, 12)$ and $Q(-8, -6)$.

Hence gradient of PQ, $m_{PQ} = \frac{-6-12}{-8-4} = \frac{-18}{-12} = \frac{3}{2}$

$$\begin{aligned} \text{Hence, } y - 12 &= \tfrac{3}{2}(x-4) \\ 2y - 24 &= 3(x-4) \\ 2y - 24 &= 3x - 12 \\ 0 &= 3x - 2y - 12 + 24 \\ 0 &= 3x - 2y + 12 \end{aligned}$$

The line PQ has equation $3x - 2y + 12 = 0$.

b From part (a), the gradient of the chord PQ is $\frac{3}{2}$.

The normal to H at A is parallel to the chord PQ, implies that the gradient of the normal to H at A is $\frac{3}{2}$.

It follows that the gradient of the tangent to H at A is

$$m_T = \frac{-1}{m_N} = \frac{-1}{\left(\frac{3}{2}\right)} = -\frac{2}{3}$$

$H : xy = 48 \Rightarrow y = 48x^{-1}$

$$\frac{dy}{dx} = -48x^{-2} = -\frac{48}{x^2}$$

At A, $m_T = \frac{dy}{dx} = -\frac{48}{x^2} = -\frac{2}{3} \Rightarrow \frac{48}{x^2} = \frac{2}{3}$

Hence, $2x^2 = 144 \Rightarrow x^2 = 72 \Rightarrow x = \pm\sqrt{72} \Rightarrow x = \pm 6\sqrt{2}$ Note: $\sqrt{72} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$.

When $x = 6\sqrt{2} \Rightarrow y = \frac{48}{6\sqrt{2}} = \frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{\sqrt{2}\sqrt{2}} = 4\sqrt{2}$.

When $x = -6\sqrt{2} \Rightarrow y = \frac{48}{-6\sqrt{2}} = \frac{-8}{\sqrt{2}} = \frac{-8\sqrt{2}}{\sqrt{2}\sqrt{2}} = -4\sqrt{2}$.

Hence the possible exact coordinates of A are $(6\sqrt{2}, 4\sqrt{2})$ or $(-6\sqrt{2}, -4\sqrt{2})$.

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Quadratic Equations
Exercise D, Question 6

Question:

The curve H is defined by the equations $x=\sqrt{3}t,\ y=\frac{\sqrt{3}}{t},\ t\in\mathbb{R},\ t\neq 0$.

The point P lies on H with x-coordinate $2\sqrt{3}$. Find:

a a Cartesian equation for the curve H,

b an equation of the normal to H at P.

The normal to H at P meets H again at the point Q.

c Find the exact coordinates of Q.

Solution:

a $xy=\sqrt{3}t\times\left(\frac{\sqrt{3}}{t}\right)$

$xy=\frac{3t}{t}$

Hence, the Cartesian equation of H is $xy=3$.

b $xy=3\Rightarrow y=3x^{-1}$

$\frac{\mathrm{d}y}{\mathrm{d}x}=-3x^{-2}=-\frac{3}{x^2}$

At $x=2\sqrt{3}$, $m_T=\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3}{(2\sqrt{3})^2}=-\frac{3}{12}=-\frac{1}{4}$

Gradient of tangent at P is $m_T=-\frac{1}{4}$.

So gradient of normal at P is $m_N=\frac{-1}{\left(-\frac{1}{4}\right)}=4$.

At P, when $x=2\sqrt{3},\Rightarrow 2\sqrt{3}=\sqrt{3}t\Rightarrow t=\frac{2\sqrt{3}}{\sqrt{3}}=2$

When $t=2, y=\frac{\sqrt{3}}{2}\Rightarrow P\left(2\sqrt{3},\frac{\sqrt{3}}{2}\right)$.

N: $y-\frac{\sqrt{3}}{2}=4(x-2\sqrt{3})$

N: $2y-\sqrt{3}=8(x-2\sqrt{3})$

N: $2y-\sqrt{3}=8x-16\sqrt{3}$

N: $0=8x-2y-16\sqrt{3}+\sqrt{3}$

N: $0 = 8x - 2y - 15\sqrt{3}$

The equation of the normal to H at P is $8x - 2y - 15\sqrt{3} = 0$.

c Normal **N:** $8x - 2y - 15\sqrt{3} = 0$ **(1)**

Hyperbola H: $xy = 3$ **(2)**

Rearranging **(2)** gives

$$y = \frac{3}{x}$$

Substituting this equation into **(1)** gives

$$8x - 2\left(\frac{3}{x}\right) - 15\sqrt{3} = 0$$

$$8x - \left(\frac{6}{x}\right) - 15\sqrt{3} = 0$$

Multiplying both sides by x gives

$$\begin{aligned} 8x - \left(\frac{6}{x}\right) - 15\sqrt{3} &= 0 \\ 8x^2 - 6 - 15\sqrt{3}x &= 0 \\ 8x^2 - 15\sqrt{3}x - 6 &= 0 \end{aligned}$$

At P, it is already known that $x = 2\sqrt{3}$, so $(x - 2\sqrt{3})$ is a factor of this quadratic equation. Hence,

$$(x - 2\sqrt{3})(8x + \sqrt{3}) = 0$$

$$x = 2\sqrt{3}\,(\text{at } P) \text{ or } x = -\frac{\sqrt{3}}{8}(\text{at } Q).$$

At P, when $x = -\frac{\sqrt{3}}{8}, \Rightarrow \frac{-\sqrt{3}}{8} = \sqrt{3}t \Rightarrow t = \frac{-\sqrt{3}}{8\sqrt{3}} = -\frac{1}{8}$

When $t = -\frac{1}{8}$, $y = \dfrac{\sqrt{3}}{\left(-\frac{1}{8}\right)} = -8\sqrt{3} \Rightarrow Q\left(-\frac{1}{8}\sqrt{3}, -8\sqrt{3}\right).$

Hence the coordinates of Q are $\left(-\frac{1}{8}\sqrt{3}, -8\sqrt{3}\right)$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise D, Question 7

Question:

The point $P(4t^2,\ 8t)$ lies on the parabola C with equation $y^2 = 16x$. The point P also lies on the rectangular hyperbola H with equation $xy = 4$.

a Find the value of t, and hence find the coordinates of P.

The normal to H at P meets the x-axis at the point N.

b Find the coordinates of N.

The tangent to C at P meets the x-axis at the point T.

c Find the coordinates of T.

d Hence, find the area of the triangle NPT.

Solution:

a Substituting $x = 4t^2$ and $y = 8t$ into $xy = 4$ gives

$(4t^2)(8t) = 4 \Rightarrow 32t^3 = 4 \Rightarrow t^3 = \frac{4}{32} = \frac{1}{8}$.

So $t = \sqrt[3]{\left(\frac{1}{8}\right)}$.

When $t = \frac{1}{2}$, $x = 4\left(\frac{1}{2}\right)^2 = 1$.

When $t = \frac{1}{2}$, $y = 8\left(\frac{1}{2}\right) = 4$.

Hence the value of t is $\frac{1}{2}$ and P has coordinates (1, 4).

b $xy = 4 \Rightarrow y = 4x^{-1}$

$\frac{dy}{dx} = -4x^{-2} = -\frac{4}{x^2}$

At $P(1, 4)$, $m_T = \frac{dy}{dx} = -\frac{4}{(1)^2} = -\frac{4}{1} = -4$

Gradient of tangent at $P(1, 4)$ is $m_T = -4$.

So gradient of normal at $P(1, 4)$ is $m_N = \frac{-1}{-4} = \frac{1}{4}$.

N: $y - 4 = \frac{1}{4}(x - 1)$

N: $4y - 16 = x - 1$

N: $0 = x - 4y + 15$

N cuts x-axis $\Rightarrow y = 0 \Rightarrow 0 = x + 15 \Rightarrow x = -15$

Therefore, the coordinates of N are $(-15, 0)$.

c For $P(1, 4)$, $y > 0$, so

$y^2 = 16x \Rightarrow y = \sqrt{16x} = \sqrt{16}\sqrt{x} = 4\sqrt{x} = 4\sqrt{x} = 4x^{\frac{1}{2}}$

So $y = 4x^{\frac{1}{2}}$

$\frac{dy}{dx} = 4\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = 2x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{2}{\sqrt{x}}$

At $P(1, 4)$, $m_T = \frac{dy}{dx} = \frac{2}{\sqrt{1}} = \frac{2}{1} = 2.$

Gradient of tangent at $P(1, 4)$ is $m_T = 2$.

T: $y - 4 = 2(x - 1)$

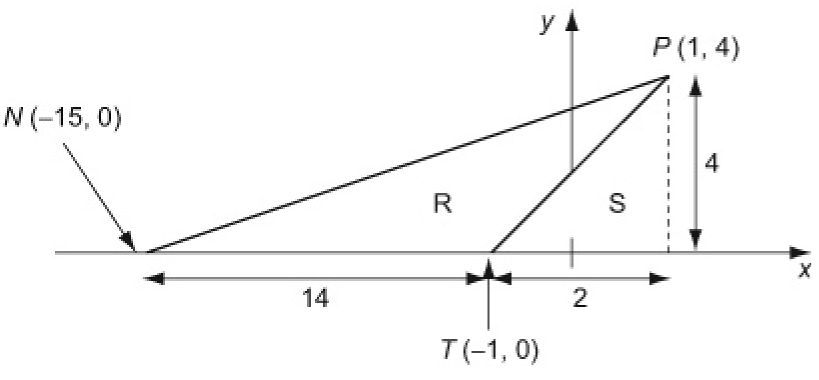
T: $y - 4 = 2x - 2$

T: $0 = 2x - y + 2$

T cuts x-axis $\Rightarrow y = 0 \Rightarrow 0 = 2x + 2 \Rightarrow x = -1$

Therefore, the coordinates of T are $(-1, 0)$.

d



Using sketch drawn, Area $\Delta\ NPT$
$$\begin{aligned} &= \text{Area}(R + S) - \text{Area}(S) \\ &= \frac{1}{2}(16)(4) - \frac{1}{2}(2)(4) \\ &= 32 - 4 \\ &= 28 \end{aligned}$$

Therefore, Area $\Delta\ NPT = 28$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise E, Question 1

Question:

The point $P(3t^2,\ 6t)$ lies on the parabola C with equation $y^2 = 12x$.

a Show that an equation of the tangent to C at P is $yt = x + 3t^2$.

b Show that an equation of the normal to C at P is $xt + y = 3t^3 + 6t$.

Solution:

a C: $y^2 = 12x \Rightarrow y = \pm\sqrt{12x} = \pm\sqrt{4}\sqrt{3}\sqrt{x} = \pm 2\sqrt{3}x^{\frac{1}{2}}$

So $y = \pm 2\sqrt{3}x^{\frac{1}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm 2\sqrt{3}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \pm\sqrt{3}x^{-\frac{1}{2}}$

So, $\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{\sqrt{3}}{\sqrt{x}}$

At $P(3t^2,\ 6t)$, $m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{\sqrt{3}}{\sqrt{3t^2}} = \pm\frac{\sqrt{3}}{\sqrt{3}t} = \frac{1}{t}$.

T: $y - 6t = \frac{1}{t}(x - 3t^2)$

T: $ty - 6t^2 = x - 3t^2$

T: $yt = x - 3t^2 + 6t^2$

T: $yt = x + 3t^2$

The equation of the tangent to C at P is $yt = x + 3t^2$.

b Gradient of tangent at $P(3t^2,\ 6t)$ is $m_T = \frac{1}{t}$.

So gradient of normal at $P(3t^2,\ 6t)$ is $m_N = \frac{-1}{\left(\frac{1}{t}\right)} = -t$.

N: $y - 6t = -t(x - 3t^2)$

N: $y - 6t = -tx + 3t^3$

N: $xt + y = 3t^3 + 6t$.

The equation of the normal to C at P is $xt + y = 3t^3 + 6t$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise E, Question 2

Question:

The point $P\left(6t,\ \frac{6}{t}\right),\ t \neq 0$, lies on the rectangular hyperbola H with equation $xy = 36$.

a Show that an equation of the tangent to H at P is $x + t^2y = 12t$.

b Show that an equation of the normal to H at P is $t^3x - ty = 6(t^4 - 1)$.

Solution:

a H: $xy = 36 \Rightarrow y = 36x^{-1}$

$\frac{dy}{dx} = -36x^{-2} = -\frac{36}{x^2}$

At $P(6t,\ \frac{6}{t}),\ m_T = \frac{dy}{dx} = -\frac{36}{(6t)^2} = -\frac{36}{36t^2} = -\frac{1}{t^2}$

T: $y - \frac{6}{t} = -\frac{1}{t^2}(x - 6t)$ (Now multiply both sides by t^2.)

T: $t^2y - 6t = -(x - 6t)$

T: $t^2y - 6t = -x + 6t$

T: $x + t^2y = 6t + 6t$

T: $x + t^2y = 12t$

The equation of the tangent to H at P is $x + t^2y = 12t$.

b Gradient of tangent at $P(6t,\ \frac{6}{t})$ is $m_T = -\frac{1}{t^2}$.

So gradient of normal at $P\left(6t,\ \frac{6}{t}\right)$ is $m_N = \frac{-1}{\left(-\frac{1}{t^2}\right)} = t^2$.

N: $y - \frac{6}{t} = t^2(x - 6t)$ (Now multiply both sides by t.)

N: $ty - 6 = t^3(x - 6t)$

N: $ty - 6 = t^3x - 6t^4$

N: $6t^4 - 6 = t^3x - ty$

N: $6(t^4 - 1) = t^3x - ty$

The equation of the normal to H at P is $t^3x - ty = 6(t^4 - 1)$.

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Quadratic Equations
Exercise E, Question 3

Question:

The point $P(5t^2,\ 10t)$ lies on the parabola C with equation $y^2 = 4ax$, where a is a constant and $t \neq 0$.

a Find the value of a.

b Show that an equation of the tangent to C at P is $yt = x + 5t^2$.

The tangent to C at P cuts the x-axis at the point X and the y-axis at the point Y. The point O is the origin of the coordinate system.

c Find, in terms of t, the area of the triangle OXY.

Solution:

a Substituting $x = 5t^2$ and $y = 10t$ into $y^2 = 4ax$ gives

$$(10t)^2 = 4(a)(5t^2) \Rightarrow 100t^2 = 20t^2 a \Rightarrow a = \frac{100t^2}{20t^2} = 5$$

So, $a = 5$.

b When $a = 5$, $y^2 = 4(5)x \Rightarrow y^2 = 20x$.

$$C\text{: } y^2 = 20x \Rightarrow y = \pm\sqrt{20x} = \pm\sqrt{4}\sqrt{5}\sqrt{x} = \pm 2\sqrt{5}\,x^{\frac{1}{2}}$$

So $y = \pm 2\sqrt{5}\,x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \pm 2\sqrt{5}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \pm\sqrt{5}\,x^{-\frac{1}{2}}$$

So, $\dfrac{dy}{dx} = \pm\dfrac{\sqrt{5}}{\sqrt{x}}$

At $P(5t^2,\ 10t)$, $m_T = \dfrac{dy}{dx} = \dfrac{\sqrt{5}}{\sqrt{5t^2}} = \dfrac{\sqrt{5}}{\sqrt{5}\,t} = \dfrac{1}{t}$.

T: $y - 10t = \dfrac{1}{t}(x - 5t^2)$

T: $ty - 10t^2 = x - 5t^2$

T: $yt = x - 5t^2 + 10t^2$

T: $yt = x + 5t^2$

Therefore, the equation of the tangent to C at P is $yt = x + 5t^2$.

For $(at^2,\ 2at)$ on $y^2 = 4ax$

We *always* get $\frac{d}{dx}(y^2) = 4a$

$2y\frac{dy}{dx} = 4a\frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{2ac} = \frac{1}{t}$

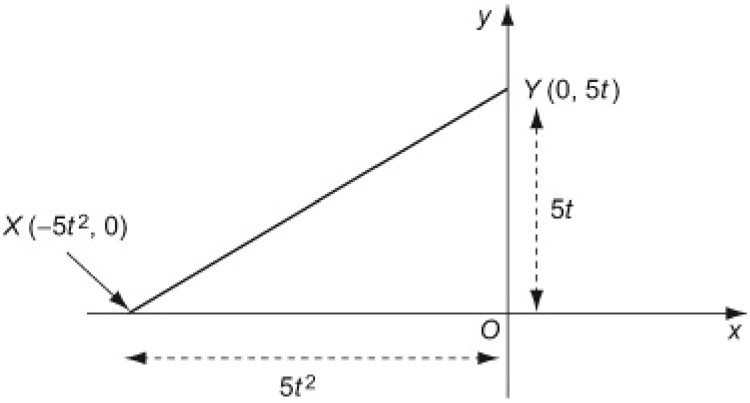
c T: $yt = x + 5t^2$

T cuts x-axis $\Rightarrow y = 0 \Rightarrow 0 = x + 5t^2 \Rightarrow x = -5t^2$

Hence the coordinates of X are $(-5t^2,\ 0)$.

T cuts y-axis $\Rightarrow x = 0 \Rightarrow yt = 5t^2 \Rightarrow y = 5t$

Hence the coordinates of Y are $(0, 5t)$.



Using sketch drawn, Area $\Delta OXY = \frac{1}{2}(5t^2)(5t)$
$= \frac{25}{2}t^3$

Therefore, Area $\Delta OXY = \frac{25}{2}t^3$

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Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise E, Question 4

Question:

The point $P(at^2,\ 2at)$, $t \neq 0$, lies on the parabola C with equation $y^2 = 4ax$, where a is a positive constant.

a Show that an equation of the tangent to C at P is $ty = x + at^2$.

The tangent to C at the point A and the tangent to C at the point B meet at the point with coordinates $(-4a,\ 3a)$.

b Find, in terms of a, the coordinates of A and the coordinates of B.

Solution:

a C: $y^2 = 4ax \Rightarrow y = \pm\sqrt{4ax} = \sqrt{4}\sqrt{a}\sqrt{x} = 2\sqrt{a}x^{\frac{1}{2}}$

So $y = 2\sqrt{a}x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\sqrt{a}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{a}x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}}$

At $P(at^2,\ 2at)$, $m_T = \frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{at^2}} = \frac{\sqrt{a}}{\sqrt{a}t} = \frac{1}{t}$.

T: $y - 2at = \frac{1}{t}(x - at^2)$

T: $ty - 2at^2 = x - at^2$

T: $ty = x - at^2 + 2at^2$

T: $ty = x + at^2$

The equation of the tangent to C at P is $ty = x + at^2$.

b As the tangent **T** goes through $(-4a,\ 3a)$, then substitute $x = -4a$ and $y = 3a$ into **T**.

$t(3a) = -4a + at^2$
$0 = at^2 - 3at - 4a$
$t^2 - 3t - 4 = 0$
$(t + 1)(t - 4) = 0$
$t = -1,\ \ 4$

When $t = -1$, $x = a(-1)^2 = a$, $y = 2a(-1) = -2a \Rightarrow (a,\ -2a)$.

When $t = 4$, $x = a(4)^2 = 16a$, $y = 2a(4) = 8a \Rightarrow (16a,\ 8a)$.

The coordinates of A and B are $(a,\ -2a)$ and $(16a,\ 8a)$.

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Quadratic Equations
Exercise E, Question 5

Question:

The point $P\left(4t,\ \frac{4}{t}\right)$, $t \neq 0$, lies on the rectangular hyperbola H with equation $xy = 16$.

a Show that an equation of the tangent to C at P is $x + t^2 y = 8t$.

The tangent to H at the point A and the tangent to H at the point B meet at the point X with y-coordinate 5. X lies on the directrix of the parabola C with equation $y^2 = 16x$.

b Write down the coordinates of X.

c Find the coordinates of A and B.

d Deduce the equations of the tangents to H which pass through X. Give your answers in the form $ax + by + c = 0$, where a, b and c are integers.

Solution:

a H: $xy = 16 \Rightarrow y = 16x^{-1}$

$\frac{dy}{dx} = -16x^{-2} = -\frac{16}{x^2}$

At $P(4t,\ \frac{4}{t})$, $m_T = \frac{dy}{dx} = -\frac{16}{(4t)^2} = -\frac{16}{16t^2} = -\frac{1}{t^2}$

T: $y - \frac{4}{t} = -\frac{1}{t^2}(x - 4t)$ (Now multiply both sides by t^2.)

T: $t^2 y - 4t = -(x - 4t)$

T: $t^2 y - 4t = -x + 4t$

T: $x + t^2 y = 4t + 4t$

T: $x + t^2 y = 8t$

The equation of the tangent to H at P is $x + t^2 y = 8t$.

b $y^2 = 16x$. So $4a = 16$, gives $a = \frac{16}{4} = 4$.

So the directrix has equation $x + 4 = 0$ or $x = -4$.

Therefore at X, $x = -4$ and as stated $y = 5$.

The coordinates of X are $(-4,\ 5)$.

c T: $x + t^2 y = 8t$

As the tangent **T** goes through $(-4,\ 5)$, then substitute $x = -4$ and $y = 5$ into **T**.

$(-4)+t^2(5)=8t$
$5t^2-4=8t$
$5t^2-8t-4=0$
$(t-2)(5t+2)=0$
$t=2,\ -\frac{2}{5}$

When $t=2$, $x=4(2)=8$, $y=\frac{4}{2}=2 \Rightarrow (8,\ 2)$.

When $t=-\frac{2}{5}$, $x=4(-\frac{2}{5})=-\frac{8}{5}$, $y=\frac{4}{\left(-\frac{2}{5}\right)}=-10 \Rightarrow (-\frac{8}{5},\ -10)$.

The coordinates of A and B are (8, 2) and $(-\frac{8}{5},\ -10)$.

d Substitute $t=2$ and $t=-\frac{2}{5}$ into **T** to find the equations of the tangents to H that go through the point X.

When $t=2$, **T:** $x+4y=16 \Rightarrow x+4y-16=0$

When $t=-\frac{2}{5}$, **T:** $x+\left(-\frac{2}{5}\right)^2 y=8\left(-\frac{2}{5}\right)$

T: $x+\frac{4}{25}y=-\frac{16}{5}$

T: $25x+4y=-80$

T: $25x+4y+80=0$

Hence the equations of the tangents are $x+4y-16=0$ and $25x+4y+80=0$.

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Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise E, Question 6

Question:

The point $P(\mathrm{at}^2,\ 2at)$ lies on the parabola C with equation $y^2 = 4ax$, where a is a constant and $t \neq 0$. The tangent to C at P cuts the x-axis at the point A.

a Find, in terms of a and t, the coordinates of A.

The normal to C at P cuts the x-axis at the point B.

b Find, in terms of a and t, the coordinates of B.

c Hence find, in terms of a and t, the area of the triangle APB.

Solution:

a C: $y^2 = 4ax \Rightarrow y = \pm\sqrt{4ax} = \sqrt{4}\sqrt{a}\sqrt{x} = 2\sqrt{a}x^{\frac{1}{2}}$

So $y = 2\sqrt{a}x^{\frac{1}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{a}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{a}x^{-\frac{1}{2}}$

So, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{a}}{\sqrt{x}}$

At $P(at^2,\ 2at)$, $m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{a}}{\sqrt{at^2}} = \frac{\sqrt{a}}{\sqrt{a}\,t} = \frac{1}{t}$.

T: $y - 2at = \frac{1}{t}(x - at^2)$

T: $ty - 2at^2 = x - at^2$

T: $ty = x - at^2 + 2at^2$

T: $ty = x + at^2$

T cuts x-axis $\Rightarrow y = 0$. So,

$0 = x + at^2 \Rightarrow x = -at^2$

The coordinates of A are $(-at^2,\ 0)$.

b Gradient of tangent at $P\left(at^2,\ 2at\right)$ is $m_T = \frac{1}{t}$.

So gradient of normal at $P\left(at^2,\ 2at\right)$ is $m_N = \frac{-1}{\left(\frac{1}{t}\right)} = -t$.

N: $y - 2at = -t(x - at^2)$

N: $y - 2at = -tx + at^3$

N cuts x-axis $\Rightarrow y = 0$. So,

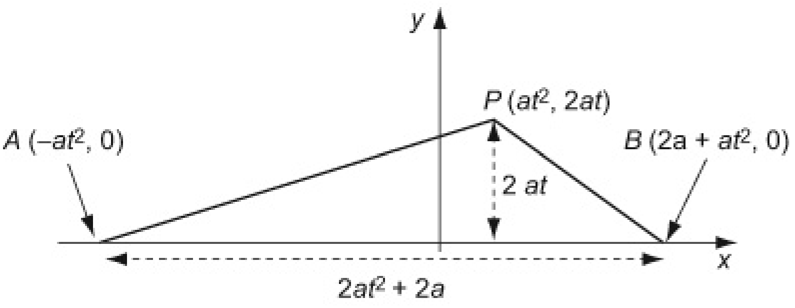
$0 - 2at = -tx + at^3$

$tx = 2at + at^3$

$x = 2a + at^2$

The coordinates of B are $(2a + at^2,\ 0)$.

c



Using sketch drawn, Area $\Delta APB = \frac{1}{2}(2a + 2at^2)(2at)$

$= at(2a + 2at^2)$

$= 2a^2t(1 + t^2)$

Therefore, Area $\Delta APB = 2a^2t(1 + t^2)$

Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Quadratic Equations

Exercise E, Question 7

Question:

The point $P(2t^2,\ 4t)$ lies on the parabola C with equation $y^2 = 8x$.

a Show that an equation of the normal to C at P is $xt + y = 2t^3 + 4t$.

The normals to C at the points R, S and T meet at the point (12, 0).

b Find the coordinates of R, S and T.

c Deduce the equations of the normals to C which all pass through the point (12, 0).

Solution:

a C: $y^2 = 8x \Rightarrow y = \pm\sqrt{8x} = \sqrt{4}\sqrt{2}\sqrt{x} = 2\sqrt{2}x^{\frac{1}{2}}$

So $y = 2\sqrt{2}x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\sqrt{2}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{2}x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{\sqrt{2}}{\sqrt{x}}$

At $P(2t^2,\ 4t)$, $m_T = \frac{dy}{dx} = \frac{\sqrt{2}}{\sqrt{2t^2}} = \frac{\sqrt{2}}{\sqrt{2}\,t} = \frac{1}{t}$.

Gradient of tangent at $P(2t^2,\ 4t)$ is $m_T = \frac{1}{t}$.

So gradient of normal at $P(2t^2,\ 4t)$ is $m_N = \frac{-1}{\left(\frac{1}{t}\right)} = -t$.

N: $y - 4t = -t(x - 2t^2)$

N: $y - 4t = -tx + 2t^3$

N: $xt + y = 2t^3 + 4t$.

The equation of the normal to C at P is $xt + y = 2t^3 + 4t$.

b As the normals go through (12, 0), then substitute $x = 12$ and $y = 0$ into **N**.

$(12)t + 0 = 2t^3 + 4t$
$12t = 2t^3 + 4t$
$0 = 2t^3 + 4t - 12t$
$0 = 2t^3 - 8t$
$t^3 - 4t = 0$
$t(t^2 - 4) = 0$
$t(t-2)(t+2) = 0$
$t = 0, \ 2, \ -2$

When $t = 0$, $x = 2(0)^2 = 0$, $y = 4(0) = 0$ $\Rightarrow (0, \ 0)$.

When $t = 2$, $x = 2(2)^2 = 8$, $y = 4(2) = 8$ $\Rightarrow (8, \ 8)$.

When $t = -2$, $x = 2(-2)^2 = 8$, $y = 4(-2) = -8$ $\Rightarrow (8, \ -8)$.

The coordinates of R, S and T are (0, 0), (8, 8) and $(8, \ -8)$.

c Substitute $t = 0, \ 2, \ -2$ into $xt + y = 2t^3 + 4t$. to find the equations of the normals to H that go through the point (12, 0).

When $t = 0$, **N:** $0 + y = 0 + 0. \Rightarrow y = 0$

When $t = 2$, **N:** $x(2) + y = 2(8) + 4(2)$

N: $2x + y = 24$

N: $2x + y - 24 = 0$

When $t = -2$, **N:** $x(-2) + y = 2(-8) + 4(-2)$

N: $-2x + y = -24$

N: $2x - y - 24 = 0$

Hence the equations of the normals are $y = 0$, $2x + y - 24 = 0$ and $2x - y - 24 = 0$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise E, Question 8

Question:

The point $P(at^2,\ 2at)$ lies on the parabola C with equation $y^2 = 4ax$, where a is a positive constant and $t \neq 0$. The tangent to C at P meets the y-axis at Q.

a Find in terms of a and t, the coordinates of Q.

The point S is the focus of the parabola.

b State the coordinates of S.

c Show that PQ is perpendicular to SQ.

Solution:

a C: $y^2 = 4ax \Rightarrow y = \sqrt{4ax} = \sqrt{4}\sqrt{a}\sqrt{x} = 2\sqrt{a}x^{\frac{1}{2}}$

So $y = 2\sqrt{a}x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\sqrt{a}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{a}x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}}$

At $P(at^2,\ 2at)$, $m_T = \frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{at^2}} = \frac{\sqrt{a}}{\sqrt{a}\,t} = \frac{1}{t}$.

T: $y - 2at = \frac{1}{t}(x - at^2)$

T: $ty - 2at^2 = x - at^2$

T: $ty = x - at^2 + 2at^2$

T: $ty = x + at^2$

T meets $y - axis \Rightarrow x = 0$. So,

$ty = 0 + at^2 \Rightarrow y = \frac{at^2}{t} \Rightarrow y = at$

The coordinates of Q are $(0,\ at)$.

b The focus of a parabola with equation $y^2 = 4ax$ has coordinates $(a,\ 0)$.

So, the coordinates of S are $(a,\ 0)$.

c $P(at^2,\ 2at)$, $Q(0,\ at)$ and $S(a,\ 0)$.

$$m_{PQ} = \frac{at - 2at}{0 - at^2} = \frac{-at}{-at^2} = \frac{1}{t}.$$

$$m_{SQ} = \frac{0 - at}{a - 0} = -\frac{at}{a} = -t.$$

Therefore, $m_{PQ} \times m_{SQ} = \frac{1}{t} \times -t = -1.$

So PQ is perpendicular to SQ.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise E, Question 9

Question:

The point $P(6t^2,\ 12t)$ lies on the parabola C with equation $y^2 = 24x$.

a Show that an equation of the tangent to the parabola at P is $ty = x + 6t^2$.

The point X has y-coordinate 9 and lies on the directrix of C.

b State the x-coordinate of X.

The tangent at the point B on C goes through point X.

c Find the possible coordinates of B.

Solution:

a C: $y^2 = 24x \Rightarrow y = \pm\sqrt{24x} = \sqrt{4}\sqrt{6}\sqrt{x} = 2\sqrt{6}x^{\frac{1}{2}}$

So $y = 2\sqrt{6}x^{\frac{1}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{6}\left(\frac{1}{2}\right)x^{\frac{1}{2}} = \sqrt{6}x^{-\frac{1}{2}}$

So, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{6}}{\sqrt{x}}$

At $P(6t^2,\ 12t)$, $m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{6}}{\sqrt{6t^2}} = \frac{\sqrt{6}}{\sqrt{6}\,t} = \frac{1}{t}$.

T: $y - 12t = \frac{1}{t}(x - 6t^2)$

T: $ty - 12t^2 = x - 6t^2$

T: $ty = x - 6t^2 + 12t^2$

T: $ty = x + 6t^2$

The equation of the tangent to C at P is $ty = x + 6t^2$.

b $y^2 = 24x$. So $4a = 24$, gives $a = \frac{24}{4} = 6$.

So the directrix has equation $x + 6 = 0$ or $x = -6$.

Therefore at X, $x = -6$.

c T: $ty = x + 6t^2$ and the coordinates of X are $(-6,\ 9)$.

As the tangent **T** goes through $(-6,\ 9)$, then substitute $x = -6$ and $y = 9$ into **T**.

$$t(9) = -6 + 6t^2$$
$$0 = 6t^2 - 9t - 6$$
$$2t^2 - 3t - 2 = 0$$
$$(t-2)(2t+1) = 0$$
$$t = 2,\ \ -\frac{1}{2}$$

When $t = 2,\quad x = 6(2)^2 = 24,\quad y = 12(2) = 24 \Rightarrow (24,\ 24).$

When $t = -\frac{1}{2},\ \ x = 6\left(-\frac{1}{2}\right)^2 = \frac{3}{2},\quad y = 12\left(-\frac{1}{2}\right) = -6 \Rightarrow \left(\frac{3}{2},\ -6\right).$

The possible coordinates of B are (24, 24) and $\left(\frac{3}{2},\ -6\right)$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise F, Question 1

Question:

A parabola C has equation $y^2 = 12x$. The point S is the focus of C.

a Find the coordinates of S.

The line l with equation $y = 3x$ intersects C at the point P where $y > 0$.

b Find the coordinates of P.

c Find the area of the triangle OPS, where O is the origin.

Solution:

a $y^2 = 12x$. So $4a = 12$, gives $a = \frac{12}{4} = 3$.

So the focus S, has coordinates (3, 0).

b Line l: $y = 3x$ **(1)**

Parabola C: $y^2 = 12x$ **(2)**

Substituting **(1)** into **(2)** gives

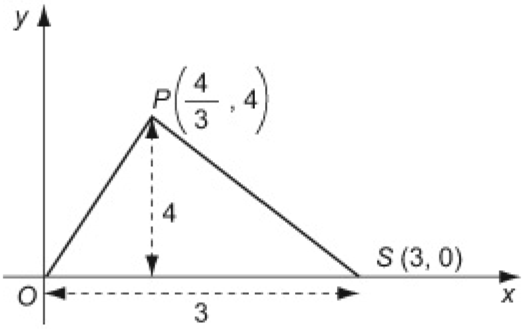
$$(3x)^2 = 12x$$
$$9x^2 = 12x$$
$$9x^2 - 12x = 0$$
$$3x(3x - 4) = 0$$
$$x = 0, \frac{4}{3}$$

Substituting these values of x back into equation **(1)** gives

$$\begin{aligned} x = 0, y = 3(0) &= 0 \Rightarrow (0, 0) \\ x = \frac{4}{3}, y = 3\left(\frac{4}{3}\right) &= 4 \Rightarrow \left(\frac{4}{3}, 4\right) \end{aligned}$$

As $y > 0$, the coordinates of P are $\left(\frac{4}{3}, 4\right)$.

c



Using sketch drawn, Area Δ *OPS* $= \frac{1}{2}(3)(4)$

$= \frac{1}{2}(12)$

$= 6$

Therefore, Area Δ *OPS* = 6

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise F, Question 2

Question:

A parabola C has equation $y^2 = 24x$. The point P with coordinates $(k, 6)$, where k is a constant lies on C.

a Find the value of k.

The point S is the focus of C.

b Find the coordinates of S.

The line l passes through S and P and intersects the directrix of C at the point D.

c Show that an equation for l is $4x + 3y - 24 = 0$.

d Find the area of the triangle OPD, where O is the origin.

Solution:

a $(k, 6)$ lies on $y^2 = 24x$ gives

$6^2 = 24k \Rightarrow 36 = 24k \Rightarrow \frac{36}{24} = k \Rightarrow k = \frac{3}{2}.$

b $y^2 = 24x$. So $4a = 24$, gives $a = \frac{24}{4} = 6$.

So the focus S, has coordinates $(6, 0)$.

c The point P and S have coordinates $P\left(\frac{3}{2}, 6\right)$ and $S(6, 0)$.

$$m_l = m_{PS} = \frac{0-6}{6-\frac{3}{2}} = \frac{-6}{\frac{9}{2}} = -\frac{12}{9} = -\frac{4}{3}$$

l: $y - 0 = -\frac{4}{3}(x - 6)$

l: $3y = -4(x - 6)$

l: $3y = -4x + 24$

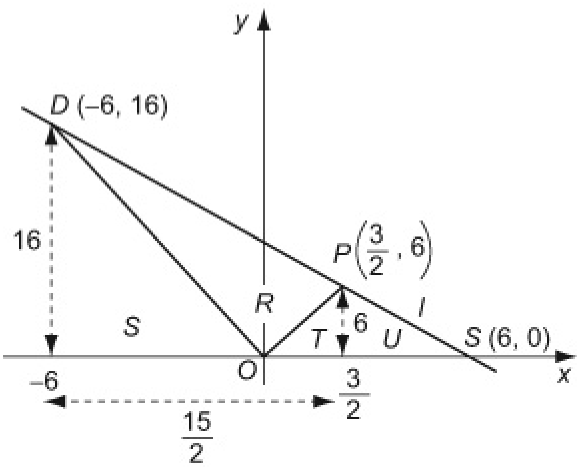
l: $4x + 3y - 24 = 0$

Therefore an equation for l is $4x + 3y - 24 = 0$.

d From (b), as $a = 6$, an equation of the directrix is $x + 6 = 0$ or $x = -6$. Substituting $x = -6$ into l gives:

$4(-6) + 3y - 24 = 0$
$3y = 24 + 24$
$3y = 48$
$y = 16$

Hence the coordinates of D are $(-6, 16)$.



Using the sketch and the regions as labeled you can find the area required. Let Area $\Delta OPD = \text{Area}(R)$

Method 1

$$\begin{aligned}
\text{Area}(R) &= \text{Area}(RST) - \text{Area}(S) - \text{Area}(T) \\
&= \frac{1}{2}(16+6)\left(\frac{15}{2}\right) - \frac{1}{2}(6)(16) - \frac{1}{2}\left(\frac{3}{2}\right)(6) \\
&= \frac{1}{2}(22)\left(\frac{15}{2}\right) - (3)(16) - \left(\frac{3}{2}\right)(3) \\
&= \left(\frac{165}{2}\right) - 48 - \left(\frac{9}{2}\right) \\
&= 30
\end{aligned}$$

Therefore, Area $\Delta OPD = 30$

Method 2

$$\begin{aligned}
\text{Area}(R) &= \text{Area}(RSTU) - \text{Area}(S) - \text{Area}(TU) \\
&= \frac{1}{2}(12)(16) - \frac{1}{2}(6)(16) - \frac{1}{2}(6)(6) \\
&= 96 - 48 - 18 \\
&= 30
\end{aligned}$$

Therefore, Area $\Delta OPD = 30$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise F, Question 3

Question:

The parabola C has parametric equations $x = 12t^2$, $y = 24t$. The focus to C is at the point S.

a Find a Cartesian equation of C.

The point P lies on C where $y > 0$. P is 28 units from S.

b Find an equation of the directrix of C.

c Find the exact coordinates of the point P.

d Find the area of the triangle OSP, giving your answer in the form $k\sqrt{3}$, where k is an integer.

Solution:

a $y = 24t$

So $t = \frac{y}{24}$ **(1)**

$x = 12t^2$ **(2)**

Substitute **(1)** into **(2)**:

$$x = 12\left(\frac{y}{24}\right)^2$$

So $x = \frac{12y^2}{576}$ simplifies to $x = \frac{y^2}{48}$

Hence, the Cartesian equation of C is $y^2 = 48x$.

b $y^2 = 48x$. So $4a = 48$, gives $a = \frac{48}{4} = 12$.

Therefore an equation of the directrix of C is $x + 12 = 0$ or $x = -12$.

c

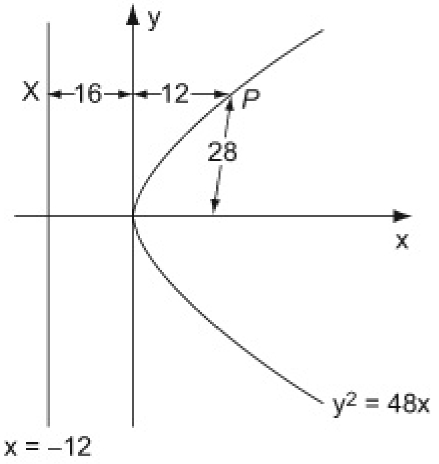
From (b), as $a = 12$, the coordinates of S, the focus to C are (12, 0). Hence, drawing a sketch gives,

The (shortest) distance of P to the line $x = -16$ is the distance XP.

The distance $SP = 28$.

The focus-directrix property implies that $SP = XP = 28$.

The directrix has equation $x = -12$.



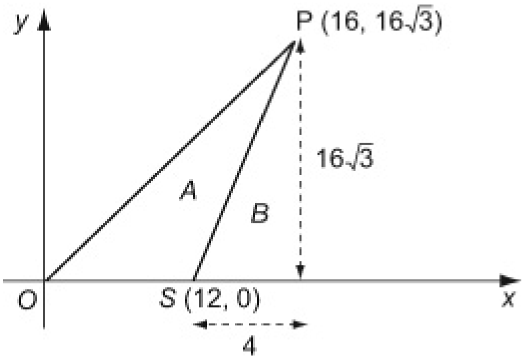
Therefore the x-coordinate of P is $x = 28 - 12 = 16$.

When $x = 16, y^2 = 48(16) \Rightarrow y^2 = 3(16)^2$

As $y > 0$, then $y = \sqrt{3(16)^2} = 16\sqrt{3}$.

Hence the exact coordinates of P are $(16, 16\sqrt{3})$.

d



Using the sketch and the regions as labeled you can find the area required. Let Area $\Delta\, OSP$ = Area(A)

$$\begin{aligned}\text{Area}(A) &= \text{Area}(AB) - \text{Area}(B)\\ &= \tfrac{1}{2}(16)(16\sqrt{3}) - \tfrac{1}{2}(4)(16\sqrt{3})\\ &= 128\sqrt{3} - 32\sqrt{3}\\ &= 96\sqrt{3}\end{aligned}$$

Therefore, Area $\Delta\, OSP = 96\sqrt{3}$ and $k = 96$.

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Quadratic Equations
Exercise F, Question 4

Question:

The point $(4t^2,\ 8t)$ lies on the parabola C with equation $y^2 = 16x$. The line l with equation $4x - 9y + 32 = 0$ intersects the curve at the points P and Q.

a Find the coordinates of P and Q.

b Show that an equation of the normal to C at $(4t^2,\ 8t)$ is $xt + y = 4t^3 + 8t$.

c Hence, find an equation of the normal to C at P and an equation of the normal to C at Q.

The normal to C at P and the normal to C at Q meet at the point R.

d Find the coordinates of R and show that R lies on C.

e Find the distance OR, giving your answer in the form $k\sqrt{97}$, where k is an integer.

Solution:

a *Method 1*

Line: $4x - 9y + 32 = 0$ **(1)**

Parabola C: $y^2 = 16x$ **(2)**

Multiplying **(1)** by 4 gives

$16x - 36y + 128 = 0$ **(3)**

Substituting **(2)** into **(3)** gives

$$y^2 - 36y + 128 = 0$$
$$(y-4)(y-32) = 0$$
$$y = 4, 32$$

When $y = 4$, $4^2 = 16x \Rightarrow x = \frac{16}{16} = 1 \Rightarrow (1, 4).$

When $y = 32$, $32^2 = 16x \Rightarrow x = \frac{1024}{16} = 64 \Rightarrow (64, 32).$

The coordinates of P and Q are (1, 4) and (64, 32).

Method 2

Line: $4x - 9y + 32 = 0$ **(1)**

Parabola C: $x = 4t^2, y = 8t$ **(2)**

Substituting **(2)** into **(1)** gives

$4(4t^2) - 9(8t) + 32 = 0$
$16t^2 - 72t + 32 = 0$
$2t^2 - 9t + 4 = 0$
$(2t - 1)(t - 4) = 0$
$t = \frac{1}{2}, 4$

When $t = \frac{1}{2}$, $x = 4\left(\frac{1}{2}\right)^2 = 1$, $y = 8\left(\frac{1}{2}\right) = 4 \Rightarrow (1, 4)$.

When $t = 4$, $x = 4(4)^2 = 64$, $y = 8(4) = 32 \Rightarrow (64, 32)$.

The coordinates of P and Q are (1, 4) and (64, 32).

b C: $y^2 = 16x \Rightarrow y = \sqrt{16x} = \sqrt{16}\sqrt{x} = 4x^{\frac{1}{2}}$

So $y = 4x^{\frac{1}{2}}$

$\frac{dy}{dx} = 4\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = 2x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{2}{\sqrt{x}}$

At $(4t^2, 8t)$, $m_T = \frac{dy}{dx} = \frac{2}{\sqrt{4t^2}} = \frac{2}{2t} = \frac{1}{t}$.

Gradient of tangent at $(4t^2, 8t)$ is $m_T = \frac{1}{t}$.

So gradient of normal at $(4t^2, 8t)$ is $m_N = \frac{-1}{\left(\frac{1}{t}\right)} = -t$.

N: $y - 8t = -t(x - 4t^2)$

N: $y - 8t = -tx + 4t^3$

N: $xt + y = 4t^3 + 8t$.

The equation of the normal to C at $(4t^2, 8t)$ is $xt + y = 4t^3 + 8t$.

c Without loss of generality, from part (a) P has coordinates (1, 4) when $t = \frac{1}{2}$ and Q has coordinates (64, 32) when $t = 4$.

When $t = \frac{1}{2}$,

N: $x\left(\frac{1}{2}\right) + y = 4\left(\frac{1}{2}\right)^3 + 8\left(\frac{1}{2}\right)$

N: $\frac{1}{2}x + y = \frac{1}{2} + 4$

N: $x + 2y = 1 + 8$

N: $x + 2y - 9 = 0$

When $t = 4$,

N: $x(4) + y = 4(4)^3 + 8(4)$

N: $4x + y = 256 + 32$

N: $4x + y - 288 = 0$

d The normals to C at P and at Q are $x + 2y - 9 = 0$ and $4x + y - 288 = 0$

N$_1$: $x + 2y - 9 = 0$ **(1)**

N$_2$: $4x + y - 288 = 0$ **(2)**

Multiplying **(2)** by 2 gives

$2 \times$ **(2) :** $8x + 2y - 576 = 0$ **(3)**

(3) $-$ **(1) :** $7x - 567 = 0$

$$\Rightarrow 7x = 567 \Rightarrow x = \frac{567}{7} = 81$$

(2) $\Rightarrow$ $y = 288 - 4(81) = 288 - 324 = -36$

The coordinates of R are $(81, -36)$.

When $y = -36$, LHS $= y^2 = (-36)^2 = 1296$

When $x = 81$, RHS $= 16x = 16(81) = 1296$

As $LHS = RHS$, R lies on C.

e The coordinates of O and R are (0, 0) and (81, −36).

$$\begin{aligned} OR &= \sqrt{(81-0)^2 + (-36-0)^2} \\ &= \sqrt{81^2 + 36^2} \\ &= \sqrt{7857} \\ &= \sqrt{(81)(97)} \\ &= \sqrt{81}\sqrt{97} \\ &= 9\sqrt{97} \end{aligned}$$

Hence the exact distance OR is $9\sqrt{97}$ and $k = 9$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise F, Question 5

Question:

The point $P(at^2,\ 2at)$ lies on the parabola C with equation $y^2 = 4ax$, where a is a positive constant. The point Q lies on the directrix of C. The point Q also lies on the x-axis.

a State the coordinates of the focus of C and the coordinates of Q.

The tangent to C at P passes through the point Q.

b Find, in terms of a, the two sets of possible coordinates of P.

Solution:

The focus and directrix of a parabola with equation $y^2 = 4ax$, are $(a, 0)$ and $x + a = 0$ respectively.

a Hence the coordinates of the focus of C are $(a, 0)$.

As Q lies on the x-axis then $y = 0$ and so Q has coordinates $(-a, 0)$.

b C: $y^2 = 4ax \Rightarrow y = \sqrt{4ax} = \sqrt{4}\sqrt{a}\sqrt{x} = 2\sqrt{a}x^{\frac{1}{2}}$

So $y = 2\sqrt{a}x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\sqrt{a}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{a}x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}}$

At $P(at^2, 2at)$, $m_T = \frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{at^2}} = \frac{\sqrt{a}}{\sqrt{a}t} = \frac{1}{t}$.

T: $y - 2at = \frac{1}{t}(x - at^2)$

T: $ty - 2at^2 = x - at^2$

T: $ty = x - at^2 + 2at^2$

T: $ty = x + at^2$

T passes through $(-a, 0)$, so substitute $x = -a,\ y = 0$ in **T**.

$t(0) = -a + at^2 \Rightarrow 0 = -a + at^2 \Rightarrow 0 = -1 + t^2$

So, $t^2 - 1 = 0 \Rightarrow (t-1)(t+1) = 0 \Rightarrow t = 1, -1$

When $t = 1,\quad x = a(1)^2 = a,\quad y = 2a(1) = 2a \quad \Rightarrow (a, 2a)$.

When $t = -1,\ x = a(-1)^2 = a,\ y = 2a(-1) = -2a \Rightarrow (a, -2a)$.

The possible coordinates of P are $(a, 2a)$ or $(a, -2a)$.

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Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise F, Question 6

Question:

The point $P\left(ct,\ \frac{c}{t}\right)$, $c > 0$, $t \neq 0$, lies on the rectangular hyperbola H with equation $xy = c^2$.

a Show that the equation of the normal to H at P is $t^3x - ty = c(t^4 - 1)$.

b Hence, find the equation of the normal n to the curve V with the equation $xy = 36$ at the point (12, 3). Give your answer in the form $ax + by = \mathrm{d}$, where a, b and d are integers.

The line n meets V again at the point Q.

c Find the coordinates of Q.

Solution:

a H: $xy = c^2 \Rightarrow y = c^2x^{-1}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -c^2x^{-2} = -\frac{c^2}{x^2}$

At $P\left(ct, \frac{c}{t}\right)$, $m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{(ct)^2} = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$

Gradient of tangent at $P\left(ct, \frac{c}{t}\right)$ is $m_T = -\frac{1}{t^2}$.

So gradient of normal at $P\left(ct, \frac{c}{t}\right)$ is $m_N = \frac{-1}{\left(-\frac{1}{t^2}\right)} = t^2$.

N: $y - \frac{c}{t} = t^2(x - ct)$ (Now multiply both sides by t.)

N: $ty - c = t^3(x - ct)$

N: $ty - c = t^3x - ct^4$

N: $ct^4 - c = t^3x - ty$

N: $t^3x - ty = ct^4 - c$

N: $t^3x - ty = c(t^4 - 1)$

The equation of the normal to H at P is $t^3x - ty = c(t^4 - 1)$.

b Comparing $xy = 36$ with $xy = c^2$ gives $c = 6$ and comparing the point (12, 3) with $\left(ct, \frac{c}{t}\right)$ gives

$ct = 12 \Rightarrow (6)t = 12 \Rightarrow t = 2$. Therefore,

n: $(2)^3x - (2)y = 6((2)^4 - 1)$

n: $8x - 2y = 6(15)$

n: $8x - 2y = 90$

n: $4x - y = 45$

An equation for n is $4x - y = 45$.

c Normal n: $4x - y = 45$ **(1)**

Hyperbola V: $xy = 36$ **(2)**

Rearranging **(2)** gives

$$y = \frac{36}{x}$$

Substituting this equation into **(1)** gives

$$4x - \left(\frac{36}{x}\right) = 45$$

Multiplying both sides by x gives

$$4x^2 - 36 = 45x$$
$$4x^2 - 45x - 36 = 0$$
$$(x - 12)(4x + 3) = 0$$
$$x = 12, -\frac{3}{4}$$

It is already known that $x = 12$. So at Q, $x = -\frac{3}{4}$.

Substituting $x = -\frac{3}{4}$ into $y = \frac{36}{x}$ gives

$$y = \frac{36}{\left(-\frac{3}{4}\right)} = -36\left(\frac{4}{3}\right) = -48.$$

Hence the coordinates of Q are $\left(-\frac{3}{4}, -48\right)$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise F, Question 7

Question:

A rectangular hyperbola H has equation $xy = 9$. The lines l_1 and l_2 are tangents to H. The gradients of l_1 and l_2 are both $-\frac{1}{4}$. Find the equations of l_1 and l_2.

Solution:

H: $xy = 9 \Rightarrow y = 9x^{-1}$

$\frac{dy}{dx} = -9x^{-2} = -\frac{9}{x^2}$

Gradients of tangent lines l_1 and l_2 are both $-\frac{1}{4}$ implies

$-\frac{9}{x^2} = -\frac{1}{4}$
$\Rightarrow x^2 = 36$
$\Rightarrow x = \pm\sqrt{36}$
$\Rightarrow x = \pm 6$

When $x = 6$, $6y = 9 \quad \Rightarrow y = \frac{9}{6} = \frac{3}{2} \quad \Rightarrow \left(6, \frac{3}{2}\right)$.

When $x = -6$, $-6y = 9 \quad \Rightarrow y = \frac{9}{-6} = -\frac{3}{2} \quad \Rightarrow \left(-6, -\frac{3}{2}\right)$.

At $\left(6, \frac{3}{2}\right)$, $m_T = -\frac{1}{4}$ and

T: $y - \frac{3}{2} = -\frac{1}{4}(x - 6)$

T: $4y - 6 = -1(x - 6)$

T: $4y - 6 = -x + 6$

T: $x + 4y - 12 = 0$

At $\left(-6, -\frac{3}{2}\right)$, $m_T = -\frac{1}{4}$ and

T: $y + \frac{3}{2} = -\frac{1}{4}(x + 6)$

T: $4y + 6 = -1(x + 6)$

T: $4y + 6 = -x - 6$

T: $x + 4y + 12 = 0$

The equations for l_1 and l_2 are $x + 4y - 12 = 0$ and $x + 4y + 12 = 0$.

Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Quadratic Equations

Exercise F, Question 8

Question:

The point P lies on the rectangular hyperbola $xy = c^2$, where $c > 0$. The tangent to the rectangular hyperbola at the point $P\left(ct,\ \frac{c}{t}\right),\ t > 0$, cuts the x-axis at the point X and cuts the y-axis at the point Y.

a Find, in terms of c and t, the coordinates of X and Y.

b Given that the area of the triangle OXY is 144, find the exact value of c.

Solution:

a H: $xy = c^2 \Rightarrow y = c^2x^{-1}$

$\frac{dy}{dx} = -c^2x^{-2} = -\frac{c^2}{x^2}$

At $P\left(ct, \frac{c}{t}\right), m_T = \frac{dy}{dx} = -\frac{c^2}{(ct)^2} = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$

T: $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$ (Now multiply both sides by t^2.)

T: $t^2y - ct = -(x - ct)$

T: $t^2y - ct = -x + ct$

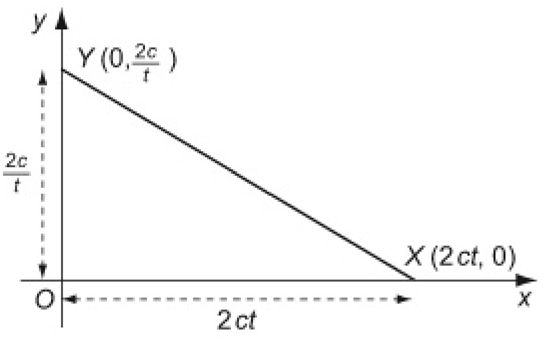
T: $x + t^2y = 2ct$

T cuts x-axis $\Rightarrow y = 0 \Rightarrow x + t^2(0) = 2ct \Rightarrow x = 2ct$

T cuts y-axis $\Rightarrow x = 0 \Rightarrow 0 + t^2y = 2ct \Rightarrow y = \frac{2ct}{t^2} = \frac{2c}{t}$

So the coordinates are $X(2ct, 0)$ and $Y\left(0, \frac{2c}{t}\right)$.

b



Using the sketch, are $\Delta\ OXY = \frac{1}{2}(2ct)\left(\frac{2c}{t}\right) = \frac{4c^2t}{2t} = 2c^2$

As area $\Delta\, OXY = 144$, then $2c^2 = 144 \Rightarrow c^2 = 72$

As $c > 0$, $c = \sqrt{72} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$.

Hence the exact value of c is $6\sqrt{2}$.

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Quadratic Equations
Exercise F, Question 9

Question:

The points $P(4at^2,\ 4at)$ and $Q(16at^2,\ 8at)$ lie on the parabola C with equation $y^2 = 4ax$, where a is a positive constant.

a Show that an equation of the tangent to C at P is $2ty = x + 4at^2$.

b Hence, write down the equation of the tangent to C at Q.

The tangent to C at P meets the tangent to C at Q at the point R.

c Find, in terms of a and t, the coordinates of R.

Solution:

a C: $y^2 = 4ax \Rightarrow y = \pm\sqrt{4ax} = \sqrt{4}\sqrt{a}\sqrt{x} = 2\sqrt{a}x^{\frac{1}{2}}$

So $y = 2\sqrt{a}x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\sqrt{a}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{a}x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}}$

At $P(4at^2, 4at)$, $m_T = \frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{4at^2}} = \frac{\sqrt{a}}{2\sqrt{a}t} = \frac{1}{2t}$.

T: $y - 4at = \frac{1}{2t}(x - 4at^2)$

T: $2ty - 8at^2 = x - 4at^2$

T: $2ty = x - 4at^2 + 8at^2$

T: $2ty = x + 4at^2$

The equation of the tangent to C at $P(4at^2, 4at)$ is $2ty = x + 4at^2$.

b P has mapped onto Q by replacing t by $2t$, ie. $t \to 2t$

So, $P(4at^2, 4at) \to Q(16at^2, 8at) = Q(4a(2t)^2, 4a(2t))$

At Q, **T** becomes $2(2t)y = x + 4a(2t)^2$

T: $2(2t)y = x + 4a(2t)^2$

T: $4ty = x + 4a(4t^2)$

T: $4ty = x + 16at^2$

The equation of the tangent to C at $Q(16at^2, 8at)$ is $4ty = x + 16at^2$.

c T_P: $2ty = x + 4at^2$ **(1)**

T_Q: $4ty = x + 16at^2$ **(2)**

(2) – (1) gives

$2ty = 12at^2$

Hence, $y = \frac{12at^2}{2t} = 6at$.

Substituting this into **(1)** gives,

$2t(6at) = x + 4at^2$

$12at^2 = x + 4at^2$

$12at^2 - 4at^2 = x$

Hence, $x = 8at^2$.

The coordinates of R are $(8at^2, 6at)$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise F, Question 10

Question:

A rectangular hyperbola H has Cartesian equation $xy = c^2$, $c > 0$. The point $\left(ct,\ \frac{c}{t}\right)$, where $t \neq 0$, $t > 0$ is a general point on H.

a Show that an equation an equation of the tangent to H at $\left(ct,\ \frac{c}{t}\right)$ is $x + t^2 y = 2ct$.

The point P lies on H. The tangent to H at P cuts the x-axis at the point X with coordinates $(2a, 0)$, where a is a constant.

b Use the answer to part **a** to show that P has coordinates $\left(a,\ \frac{c^2}{a}\right)$.

The point Q, which lies on H, has x-coordinate $2a$.

c Find the y-coordinate of Q.

d Hence, find the equation of the line OQ, where O is the origin.

The lines OQ and XP meet at point R.

e Find, in terms of a, the x-coordinate of R.

Given that the line OQ is perpendicular to the line XP,

f Show that $c^2 = 2a^2$,

g find, in terms of a, the y-coordinate of R.

Solution:

a H: $xy = c^2 \Rightarrow y = c^2 x^{-1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -c^2 x^{-2} = -\frac{c^2}{x^2}$$

At $\left(ct, \frac{c}{t}\right)$, $m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{(ct)^2} = -\frac{c^2}{c^2 t^2} = -\frac{1}{t^2}$

T: $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$ (Now multiply both sides by t^2.)

T: $t^2 y - ct = -(x - ct)$

T: $t^2 y - ct = -x + ct$

T: $x + t^2 y = 2ct$

An equation of a tangent to H at $\left(ct, \frac{c}{t}\right)$ is $x + t^2 y = 2ct$.

b **T** passes through $X(2a, 0)$, so substitute $x = 2a, y = 0$ into **T**.

$(2a) + t^2(0) = 2ct \Rightarrow 2a = 2ct \Rightarrow \frac{2a}{2c} = t \Rightarrow t = \frac{a}{c}$

Substitute $t = \frac{a}{c}$ into $\left(ct, \frac{c}{t}\right)$ gives

$$P\left(c\left(\frac{a}{c}\right), \frac{c}{\left(\frac{a}{c}\right)}\right) = P\left(a, \frac{c^2}{a}\right).$$

Hence P has coordinates $P\left(a, \frac{c^2}{a}\right)$.

c Substituting $x = 2a$ into the curve H gives

$(2a)y = c^2 \Rightarrow y = \frac{c^2}{2a}$.

The y-coordinate of Q is $y = \frac{c^2}{2a}$.

d The coordinates of O and Q are $(0, 0)$ and $\left(2a, \frac{c^2}{2a}\right)$.

$$m_{OQ} = \frac{\frac{c^2}{2a} - 0}{2a - 0} = \frac{c^2}{2a(2a)} = \frac{c^2}{4a^2}$$

OQ: $y - 0 = \frac{c^2}{4a^2}(x - 0)$

OQ: $y = \frac{c^2x}{4a^2}$. **(1)**

The equation of OQ is $y = \frac{c^2x}{4a^2}$.

e The coordinates of X and P are $(2a, 0)$ and $\left(a, \frac{c^2}{a}\right)$.

$$m_{XP} = \frac{\frac{c^2}{a} - 0}{a - 2a} = \frac{\frac{c^2}{a}}{-a} = -\frac{c^2}{a^2}$$

XP: $y - 0 = -\frac{c^2}{a^2}(x - 2a)$

XP: $y = -\frac{c^2}{a^2}(x - 2a)$ **(2)**

Substituting **(1)** into **(2)** gives,

$$\frac{c^2x}{4a^2} = -\frac{c^2}{a^2}(x - 2a)$$

Cancelling $\frac{c^2}{a^2}$ gives,

$\frac{x}{4} = -(x - 2a)$

$\frac{x}{4} = -x + 2a$

$\frac{5x}{4} = 2a$

$x = \frac{4(2a)}{5} = \frac{8a}{5}$

The x-coordinate of R is $\frac{8a}{5}$.

f From earlier parts, $m_{OQ} = \frac{c^2}{4a^2}$ and $m_{XP} = -\frac{c^2}{a^2}$

OP is perpendicular to $XP \Rightarrow m_{OQ} \times m_{XP} = -1$, gives

$m_{OQ} \times m_{XP} = \left(\frac{c^2}{4a^2}\right)\left(-\frac{c^2}{a^2}\right) = \frac{-c^4}{4a^4} = -1$

$-c^4 = -4a^4 \Rightarrow c^4 = 4a^4 \Rightarrow \left(c^2\right)^2 = 4a^4$

$c^2 = \sqrt{4a^4} = \sqrt{4}\sqrt{a^4} = 2a^2$.

Hence, $c^2 = 2a^2$, as required.

g At $R, x = \frac{8a}{5}$. Substituting $x = \frac{8a}{5}$ into $y = \frac{c^2 x}{4a^2}$ gives,

$y = \frac{c^2}{4a^2}\left(\frac{8a}{5}\right) = \frac{8ac^2}{20a^2}$

and using the $c^2 = 2a^2$ gives,

$y = \frac{8a(2a^2)}{20a^2} = \frac{16a^3}{20a^2} = \frac{4a}{5}$.

The y-coordinate of R is $\frac{4a}{5}$.

Solutionbank FP1
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Matrix algebra
Exercise A, Question 1

Question:

Describe the dimensions of these matrices.

a $\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}$

b $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

c $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \end{pmatrix}$

d $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

e $\begin{pmatrix} 3 & -1 \end{pmatrix}$

f $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution:

a $\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}$ is 2×2

b $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is 2×1

c $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \end{pmatrix}$ is 2×3

d $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ is 1×3

e $\begin{pmatrix} 3 & -1 \end{pmatrix}$ is 1×2

f $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is 3×3

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Matrix algebra
Exercise A, Question 2

Question:

For the matrices

$\mathbf{A}=\begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}4 & 1\\-1 & -2\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}6 & 0\\0 & 1\end{pmatrix}$,

find

a $\mathbf{A}+\mathbf{C}$

b $\mathbf{B}-\mathbf{A}$

c $\mathbf{A}+\mathbf{B}-\mathbf{C}$.

Solution:

a $\begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}+\begin{pmatrix}6 & 0\\0 & 1\end{pmatrix}=\begin{pmatrix}8 & -1\\1 & 4\end{pmatrix}$

b $\begin{pmatrix}4 & 1\\-1 & -2\end{pmatrix}-\begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}=\begin{pmatrix}2 & 2\\-2 & -5\end{pmatrix}$

c $\begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}+\begin{pmatrix}4 & 1\\-1 & -2\end{pmatrix}-\begin{pmatrix}6 & 0\\0 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Matrix algebra
Exercise A, Question 3

Question:

For the matrices

$\mathbf{A}=\begin{pmatrix}1\\2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & -1\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}-1 & 1 & 0\end{pmatrix}$,

$\mathbf{D}=\begin{pmatrix}0 & 1 & -1\end{pmatrix}$, $\mathbf{E}=\begin{pmatrix}3\\-1\end{pmatrix}$, $\mathbf{F}=\begin{pmatrix}2 & 1 & 3\end{pmatrix}$,

find where possible:

a $\mathbf{A}+\mathbf{B}$

b $\mathbf{A}-\mathbf{E}$

c $\mathbf{F}-\mathbf{D}+\mathbf{C}$

d $\mathbf{B}+\mathbf{C}$

e $\mathbf{F}-(\mathbf{D}+\mathbf{C})$

f $\mathbf{A}-\mathbf{F}$

g $\mathbf{C}-(\mathbf{F}-\mathbf{D})$.

Solution:

a $A+B$ is $(2\times 1)+(1\times 2)$ Not possible

b $A-E=\begin{pmatrix}1\\2\end{pmatrix}-\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}-2\\3\end{pmatrix}$.

c

$$\begin{aligned}F-D+C &= \begin{pmatrix}2 & 1 & 3\end{pmatrix}-\begin{pmatrix}0 & 1 & -1\end{pmatrix}+\begin{pmatrix}-1 & 1 & 0\end{pmatrix}\\ &= \begin{pmatrix}1 & 1 & 4\end{pmatrix}\end{aligned}$$

d $B+C$ is $(1\times 2)+(1\times 3)$ Not possible

e

$$\begin{aligned}F-(D+C) &= \begin{pmatrix}2 & 1 & 3\end{pmatrix}-[\begin{pmatrix}0 & 1 & -1\end{pmatrix}+\begin{pmatrix}-1 & 1 & 0\end{pmatrix}]\\ &= \begin{pmatrix}2 & 1 & 3\end{pmatrix}-\begin{pmatrix}-1 & 2 & -1\end{pmatrix}\\ &= \begin{pmatrix}3 & -1 & 4\end{pmatrix}\end{aligned}$$

f $A-F=(2\times 1)-(1\times 3)$ Not possible.

g

$$\begin{aligned}C-(F-D) &= \begin{pmatrix}-1 & 1 & 0\end{pmatrix}-[\begin{pmatrix}2 & 1 & 3\end{pmatrix}-\begin{pmatrix}0 & 1 & -1\end{pmatrix}]\\ &= \begin{pmatrix}-1 & 1 & 0\end{pmatrix}-\begin{pmatrix}2 & 0 & 4\end{pmatrix}\\ &= \begin{pmatrix}-3 & 1 & -4\end{pmatrix}\end{aligned}$$

Solutionbank FP1
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Matrix algebra
Exercise A, Question 4

Question:

Given that $\begin{pmatrix} a & 2 \\ -1 & b \end{pmatrix} - \begin{pmatrix} 1 & c \\ d & -2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$, find the values of the constants a, b, c and d.

Solution:

$$\begin{aligned} a - 1 &= 5 \Rightarrow a = 6 \\ 2 - c &= 0 \Rightarrow c = 2 \\ -1 - d &= 0 \Rightarrow d = -1 \\ b - (-2) &= 5 \Rightarrow b = 3 \end{aligned}$$

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Matrix algebra
Exercise A, Question 5

Question:

Given that $\begin{pmatrix} 1 & 2 & 0 \\ a & b & c \end{pmatrix} + \begin{pmatrix} a & b & c \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} c & 5 & c \\ c & c & c \end{pmatrix}$, find the values of a, b and c.

Solution:

$$\begin{aligned} 1+a &= c \quad ① \\ 2+b &= 5 \quad \Rightarrow \quad b=3 \\ 0+c &= c \\ a+1 &= c \\ b+2 &= c \quad ② \\ c+0 &= c \end{aligned}$$

Use $b=3$ in ② $\Rightarrow$ $c=5$

Use $c=5$ in ① $\Rightarrow$ $a=4$

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Matrix algebra
Exercise A, Question 6

Question:

Given that $\begin{pmatrix} 5 & 3 \\ 0 & -1 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ 2 & 0 \\ 1 & 4 \end{pmatrix}$, find the values of a, b, c, d, e and f.

Solution:

$$\begin{aligned} 5+a &= 7 && \Rightarrow a = 2 \\ 3+b &= 1 && \Rightarrow b = -2 \\ 0+c &= 2 && \Rightarrow c = 2 \\ -1+d &= 0 && \Rightarrow d = 1 \\ 2+e &= 1 && \Rightarrow e = -1 \\ 1+f &= 4 && \Rightarrow f = 3 \end{aligned}$$

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Matrix algebra
Exercise B, Question 1

Question:

For the matrices $\mathbf{A}=\begin{pmatrix}2 & 0\\4 & -6\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1\\-1\end{pmatrix}$, find

a $3\mathbf{A}$

b $\frac{1}{2}\mathbf{A}$

c $2\mathbf{B}$.

Solution:

a $3\begin{pmatrix}2 & 0\\4 & -6\end{pmatrix}=\begin{pmatrix}6 & 0\\12 & -18\end{pmatrix}$

b $\frac{1}{2}\begin{pmatrix}2 & 0\\4 & -6\end{pmatrix}=\begin{pmatrix}1 & 0\\2 & -3\end{pmatrix}$

c $2\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}2\\-2\end{pmatrix}$

Solutionbank FP1

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Matrix algebra

Exercise B, Question 2

Question:

Find the value of k and the value of x so that $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + k\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 7 \\ x & 0 \end{pmatrix}$.

Solution:

$$\begin{aligned} & 1+2k &=& \ 7 \\ \Rightarrow & \quad 2k &=& \ 6 \\ \Rightarrow & \quad k &=& \ 3 \\ & 2-k &=& \ x \\ \Rightarrow & \quad 2-3 &=& \ x \\ \therefore & \quad x &=& -1 \end{aligned}$$

Solutionbank FP1
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Matrix algebra
Exercise B, Question 3

Question:

Find the values of a, b, c and d so that $2\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} - 3\begin{pmatrix} 1 & c \\ d & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -4 & -4 \end{pmatrix}$.

Solution:

$$\begin{aligned}
2a - 3 = 3 \quad &\Rightarrow \quad 2a = 6 \\
&\Rightarrow \quad a = 3 \\
0 - 3c = 3 \quad &\Rightarrow \quad c = -1 \\
2 - 3d = -4 \quad &\Rightarrow \quad -3d = -6 \\
&\Rightarrow \quad d = 2 \\
2b + 3 = -4 \quad &\Rightarrow \quad 2b = -7 \\
&\Rightarrow \quad b = -3.5
\end{aligned}$$

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Matrix algebra
Exercise B, Question 4

Question:

Find the values of a, b, c and d so that $\begin{pmatrix} 5 & a \\ b & 0 \end{pmatrix} - 2\begin{pmatrix} c & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 1 \\ 3 & d \end{pmatrix}$.

Solution:

$$
\begin{aligned}
& 5 - 2c = 9 \\
\Rightarrow \quad & -4 = 2c \\
\Rightarrow \quad & c = -2 \\
& a - 4 = 1 \\
\Rightarrow \quad & a = 5 \\
& b - 2 = 3 \\
\Rightarrow \quad & b = 5 \\
& 0 + 2 = d \\
\Rightarrow \quad & d = 2
\end{aligned}
$$

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Matrix algebra
Exercise B, Question 5

Question:

Find the value of k so that $\begin{pmatrix}-3\\k\end{pmatrix}+k\begin{pmatrix}2k\\2k\end{pmatrix}=\begin{pmatrix}k\\6\end{pmatrix}$.

Solution:

$$
\begin{aligned}
& & -3+2k^2 &= k \\
&\Longrightarrow & 2k^2-k-3 &= 0 \\
& & (2k-3)(k+1) &= 0 \\
&\therefore & k &= \tfrac{3}{2} \text{ or } -1 \\
&\text{AND} & k+2k^2 &= 6 \\
&\Longrightarrow & 2k^2+k-6 &= 0 \\
& & (2k-3)(k+2) &= 0 \\
&\therefore & k &= \tfrac{3}{2} \text{ or } -2
\end{aligned}
$$

So common value is $k=\frac{3}{2}$

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Matrix algebra
Exercise C, Question 1

Question:

Given the dimensions of the following matrices:

Matrix	**A**	**B**	**C**	**D**	**E**
Dimension	2 × 2	1 × 2	1 × 3	3 × 2	2 × 3

Give the dimensions of these matrix products.

a BA

b DE

c CD

d ED

e AE

f DA

Solution:

a $(1 \times 2) \cdot (2 \times 2) = 1 \times 2$

b $(3 \times 2) \cdot (2 \times 3) = 3 \times 3$

c $(1 \times 3) \cdot (3 \times 2) = 1 \times 2$

d $(2 \times 3) \cdot (3 \times 2) = 2 \times 2$

e $(2 \times 2) \cdot (2 \times 3) = 2 \times 3$

f $(3 \times 2) \cdot (2 \times 2) = 3 \times 2$

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Matrix algebra
Exercise C, Question 2

Question:

Find these products.

a $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

b $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ -1 & -2 \end{pmatrix}$

Solution:

a $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

b $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -4 & 7 \end{pmatrix}$

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Matrix algebra
Exercise C, Question 3

Question:

The matrix **A**= $\begin{pmatrix} -1 & -2 \\ 0 & 3 \end{pmatrix}$ and the matrix B= $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Find

A^2 means **A** × **A**

a AB

b A^2

Solution:

a $\begin{pmatrix} -1 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & -2 & -1 \\ 3 & 3 & 0 \end{pmatrix}$

b $\begin{pmatrix} -1 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 0 & 9 \end{pmatrix}$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Matrix algebra
Exercise C, Question 4

Question:

The matrices **A, B** and **C** are given by

$\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -3 & -2 \end{pmatrix}$

Determine whether or not the following products are possible and find the products of those that are.

a AB

b AC

c BC

d BA

e CA

f CB

Solution:

a AB is $(2 \times 1) \cdot (2 \times 2)$ Not possible

b $\mathbf{AC} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} -3 & -2 \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ -3 & -2 \end{pmatrix}$

c BC is $(2 \times 2) \cdot (1 \times 2)$ Not possible

d $\mathbf{BA} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$

e $\mathbf{CA} = \begin{pmatrix} -3 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \end{pmatrix}$.

f $\mathbf{CB} = \begin{pmatrix} -3 & -2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -7 & -7 \end{pmatrix}$

Solutionbank FP1
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Matrix algebra
Exercise C, Question 5

Question:

Find in terms of a $\begin{pmatrix} 2 & a \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix}$.

Solution:

$$\begin{pmatrix} 2 & a \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 6-a & 2a \\ 1 & 4 & -2 \end{pmatrix}$$

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Matrix algebra
Exercise C, Question 6

Question:

Find in terms of x $\begin{pmatrix} 3 & 2 \\ -1 & x \end{pmatrix}\begin{pmatrix} x & -2 \\ 1 & 3 \end{pmatrix}$.

Solution:

$$\begin{pmatrix} 3 & 2 \\ -1 & x \end{pmatrix}\begin{pmatrix} x & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3x+2 & 0 \\ 0 & 3x+2 \end{pmatrix}$$

Solutionbank FP1
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Matrix algebra
Exercise C, Question 7

Question:

The matrix **A**= $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Find

a $\mathbf{A}^2$

b $\mathbf{A}^3$

c Suggest a form for $\mathbf{A}^k$.

You might be asked to prove this formula for $\mathbf{A}^k$ in FP1 using induction from Chapter 6.

Solution:

a $A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

b $A^3 = AA^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$

Note $A^2 = \begin{pmatrix} 1 & 2\times 2 \\ 0 & 1 \end{pmatrix}$

$A^3 = \begin{pmatrix} 1 & 2\times 3 \\ 0 & 1 \end{pmatrix}$

Suggests $A^k = \begin{pmatrix} 1 & 2\times k \\ 0 & 1 \end{pmatrix}$

Solutionbank FP1
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Matrix algebra
Exercise C, Question 8

Question:

The matrix **A**= $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$.

a Find, in terms of a and b, the matrix $\mathbf{A}^2$.

Given that $\mathbf{A}^2 = 3\mathbf{A}$

b find the value of a.

Solution:

a $\mathrm{A}^2 = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ ab & 0 \end{pmatrix}$

b $\mathrm{A}^2 = 3\,\mathrm{A} \Rightarrow \begin{pmatrix} a^2 & 0 \\ ab & 0 \end{pmatrix} = \begin{pmatrix} 3a & 0 \\ 3b & 0 \end{pmatrix}$

$\Rightarrow \quad a^2 = 3a \quad \Rightarrow \quad a = 3 \quad (\text{or } 0)$
and $\quad ab = 3b \quad \Rightarrow \quad a = 3$
$\therefore \quad a = 3$

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Matrix algebra
Exercise C, Question 9

Question:

$\mathbf{A} = \begin{pmatrix} -1 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix}$.

Find **a BAC**

b $\mathbf{AC}^2$

Solution:

a

$$\begin{aligned} \mathrm{BAC} &= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 3 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -4 & -7 \end{pmatrix} \\ &= \begin{pmatrix} -8 & -14 \\ -4 & -7 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

b

$$\begin{aligned} \mathrm{AC}^2 &= \begin{pmatrix} -1 & 3 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix} \\ &= \begin{pmatrix} -4 & -7 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix} \\ &= \begin{pmatrix} -16 & 29 \end{pmatrix} \end{aligned}$$

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Matrix algebra
Exercise C, Question 10

Question:

A$=\begin{pmatrix}1\\-1\\2\end{pmatrix}$, **B**$=\begin{pmatrix}3 & -2 & -3\end{pmatrix}$.

Find **a ABA**

b BAB

Solution:

a

$$\begin{aligned}\text{ABA} &= \begin{pmatrix}1\\-1\\2\end{pmatrix}\begin{pmatrix}3 & -2 & -3\end{pmatrix}\begin{pmatrix}1\\-1\\2\end{pmatrix}\\ &= \begin{pmatrix}1\\-1\\2\end{pmatrix}(-1)\\ &= \begin{pmatrix}-1\\1\\-2\end{pmatrix}\end{aligned}$$

b

$$\begin{aligned}\text{BAB} &= \begin{pmatrix}3 & -2 & -3\end{pmatrix}\begin{pmatrix}1\\-1\\2\end{pmatrix}\begin{pmatrix}3 & -2 & -3\end{pmatrix}\\ &= (-1)\begin{pmatrix}3 & -2 & -3\end{pmatrix}\\ &= \begin{pmatrix}-3 & 2 & 3\end{pmatrix}\end{aligned}$$

Solutionbank FP1
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Matrix algebra
Exercise D, Question 1

Question:

Which of the following are not linear transformations?

a **P**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2x \\ y+1 \end{pmatrix}$

b **Q**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x^2 \\ y \end{pmatrix}$

c **R**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2x+y \\ x+xy \end{pmatrix}$

d **S**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 3y \\ -x \end{pmatrix}$

e **T**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y+3 \\ x+3 \end{pmatrix}$

f **U**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2x \\ 3y-2x \end{pmatrix}$

Solution:

a P is not $\because$ $(0,0) \to (0,1)$

b Q is not $\because$ $x \to x^2$ is not linear

c R is not $\because$ $y \to x+xy$ is not linear

d S is linear

e T is not $\because$ $(0,0) \to (3,3)$

f U is linear.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Matrix algebra
Exercise D, Question 2

Question:

Identify which of these are linear transformations and give their matrix representations. Give reasons to explain why the other transformations are not linear.

a **S**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2x - y \\ 3x \end{pmatrix}$

b **T**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2y + 1 \\ x - 1 \end{pmatrix}$

c **U**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} xy \\ 0 \end{pmatrix}$

d **V**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2y \\ -x \end{pmatrix}$

e **W**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y \\ x \end{pmatrix}$

Solution:

a S is represented by $\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$

b T is not linear $\because$ $(0,0) \to (1,-1)$

c U is not linear $\because$ $x \to xy$ is not linear

d V is represented by $\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$

e W is represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Matrix algebra
Exercise D, Question 3

Question:

Identify which of these are linear transformations and give their matrix representations. Give reasons to explain why the other transformations are not linear.

a **S**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$

b **T**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ x \end{pmatrix}$

c **U**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x-y \\ x-y \end{pmatrix}$

d **V**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

e **W**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix}$

Solution:

a S is not linear $\because$ $x \to x^2$ and $y \to y^2$ are not linear

b T is represented by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

c U is represented by $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

d V is represented by $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

e W is represented by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Matrix algebra
Exercise D, Question 4

Question:

Find matrix representations for these linear transformations.

a $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y+2x \\ -y \end{pmatrix}$

b $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ x+2y \end{pmatrix}$

Solution:

a $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y+2x \\ -y \end{pmatrix} = \begin{pmatrix} 2x+y \\ 0x-y \end{pmatrix}$ is represented by $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$

b $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0-y \\ x+2y \end{pmatrix}$ is represented by $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Matrix algebra
Exercise D, Question 5

Question:

The triangle T has vertices at $(-1,\ 1)$, $(2,\ 3)$ and $(5, 1)$.

Find the vertices of the image of T under the transformations represented by these matrices.

a $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} 1 & 4 \\ 0 & -2 \end{pmatrix}$

c $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

Solution:

a $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 & 5 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -5 \\ 1 & 3 & 1 \end{pmatrix}$

$\therefore$ vertices of image of T are at $(1,1)$; $(-1,3)$; $(-5,1)$

b $\begin{pmatrix} 1 & 4 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} -1 & 2 & 5 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 14 & 9 \\ -2 & -6 & -2 \end{pmatrix}$

$\therefore$ vertices of image of T are at $(3,-2)$; $(14,-6)$; $(9,-2)$

c $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 & 5 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -6 & -2 \\ -2 & 4 & 10 \end{pmatrix}$

$\therefore$ vertices of image of T are at $(-2,-2)$; $(-6,4)$; $(-2,10)$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Matrix algebra
Exercise D, Question 6

Question:

The square S has vertices at $(-1,\ 0)$, $(0,\ 1)$, $(1,\ 0)$ and $(0\ ,\ -1)$.

Find the vertices of the image of S under the transformations represented by these matrices.

a $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

b $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

c $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Solution:

a $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 \end{pmatrix}$

$\therefore$ vertices of the image of S are $(-2,0)$: $(0,3)$; $(2,0)$; $(0\ ,-3)$

b $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$

$\therefore$ vertices of the image of S are $(-1\ ,-1)$; $(-1,1)$; $(1,1)$; $(1\ ,-1)$

c $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$

$\therefore$ vertices of the image of S are $(-1\ ,-1)$; $(1\ ,-1)$; $(1,1)$; $(-1,1)$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Matrix algebra
Exercise E, Question 1

Question:

Describe fully the geometrical transformations represented by these matrices.

a $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

b $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

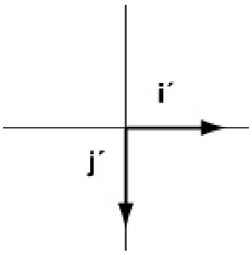
c $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Solution:

a

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

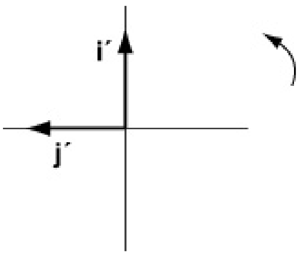


Reflection is x-axis (or line $y = 0$)

b

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

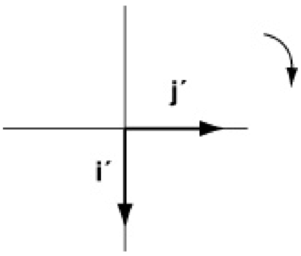


Rotation 90° anticlockwise about (0,0)

c

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



Rotation 90° clockwise (or 270° anticlockwise) about (0,0)

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Matrix algebra
Exercise E, Question 2

Question:

Describe fully the geometrical transformations represented by these matrices.

a $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

b $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

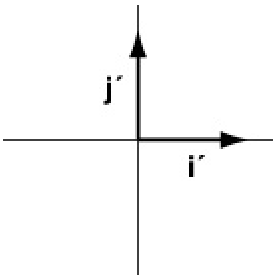
c $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution:

a

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$

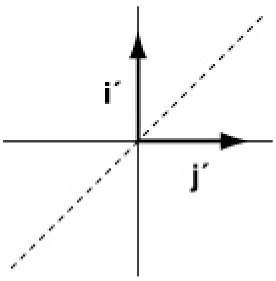


Enlargement - scale factor $\frac{1}{2}$ centre (0,0)

b

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

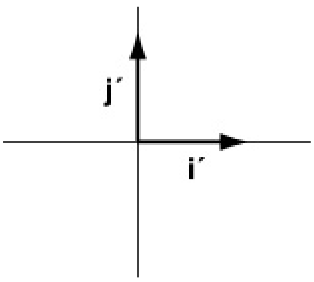


Reflection in line $y = x$

c

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



No change so this is the Identity matrix.

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Matrix algebra
Exercise E, Question 3

Question:

Describe fully the geometrical transformations represented by these matrices.

a $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

b $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

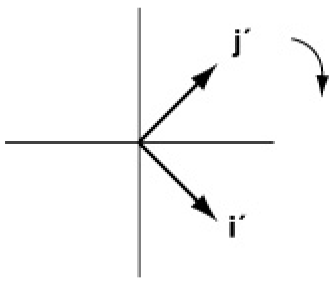
c $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$

Solution:

a

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

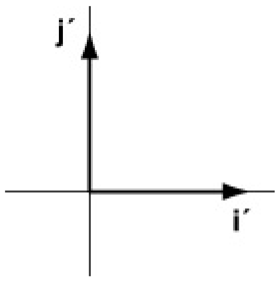


Rotation 45° clockwise about (0,0)

b

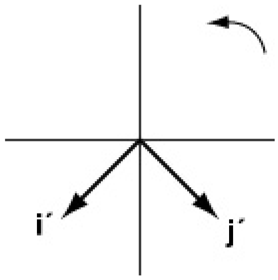
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$



Enlargement Scale factor 4 centre (0,0)

c



$$\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}-\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}$$

Rotation 225° anti-clockwise about (0,0) or 135° clockwise

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Matrix algebra
Exercise E, Question 4

Question:

Find the matrix that represents these transformations.

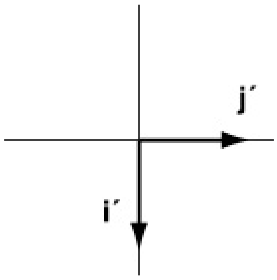
a Rotation of 90° clockwise about (0, 0).

b Reflection in the *x*-axis.

c Enlargement centre (0, 0) scale factor 2.

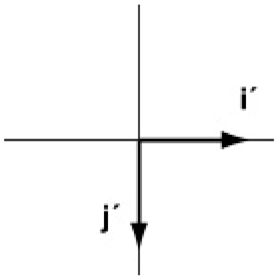
Solution:

a



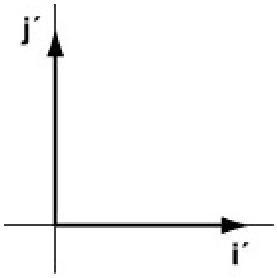
$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}0\\-1\end{pmatrix}$$
$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix}$$
$\therefore$ Matrix is $\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$

b



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix}$$
$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\-1\end{pmatrix}$$
$\therefore$ Matrix is $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

c



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}2\\0\end{pmatrix}$$
$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\2\end{pmatrix}$$
$\therefore$ Matrix is $\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$

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Matrix algebra
Exercise E, Question 5

Question:

Find the matrix that represents these transformations.

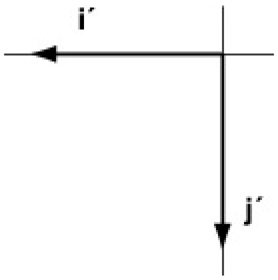
a Enlargement scale factor –4 centre (0, 0).

b Reflection in the line $y = x$.

c Rotation about (0, 0) of 135° anticlockwise.

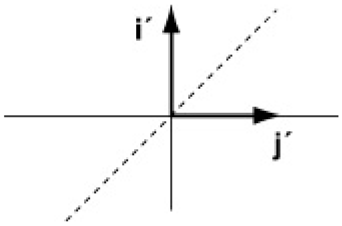
Solution:

a



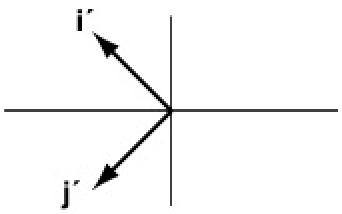
$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}-4\\0\end{pmatrix}$$
$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\-4\end{pmatrix}$$
$$\therefore \text{ Matrix is } \begin{pmatrix}-4 & 0\\0 & -4\end{pmatrix}$$

b



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}0\\1\end{pmatrix}$$
$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix}$$
$$\therefore \text{ Matrix is } \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

c



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$
$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$$
$$\therefore \text{ Matrix is } \begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$$

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Matrix algebra
Exercise F, Question 1

Question:

A= $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, **B**= $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, **C**= $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Find these matrix products and describe the single transformation represented by the product.

a AB

b BA

c AC

d A^2

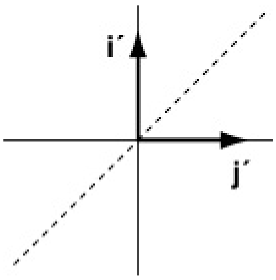
e C^2

Solution:

a AB = $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



Reflection in $y = x$

b BA = $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Reflection in $y = x$

c AC = $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ Enlargement scale facter -2 centre $(0,0)$

d $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Identity (No transformation)

[This can be thought of as a rotation of $180^\circ + 180^\circ = 360^\circ$]

e $C^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

Enlargement scale facter 4 centre (0,0)

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Matrix algebra
Exercise F, Question 2

Question:

A = rotation of 90° anticlockwise about (0, 0) B = rotation of 180° about (0, 0)
C = reflection in the x-axis D = reflection in the y-axis

a Find matrix representations of each of the four transformations A, B, C and D.

b Use matrix products to identify the single geometric transformation represented by each of these combinations.

i Reflection in the x-axis followed by a rotation of 180° about (0, 0).

ii Rotation of 180° about (0, 0) followed by a reflection in the x-axis.

iii Reflection in the y-axis followed by reflection in the x-axis.

iv Reflection in the y-axis followed by rotation of 90° about (0, 0).

v Rotation of 180° about (0, 0) followed by a second rotation of 180° about (0, 0).

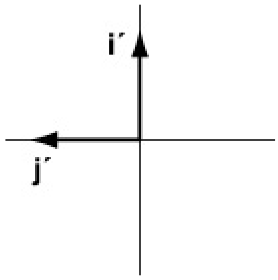
vi Reflection in the x-axis followed by rotation of 90° about (0, 0) followed by a reflection in the y-axis.

vii Reflection in the y-axis followed by rotation of 180° about (0, 0) followed by a reflection in the x-axis.

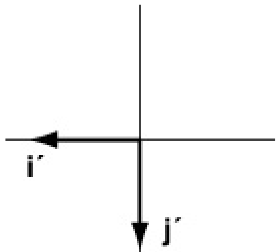
Solution:

a

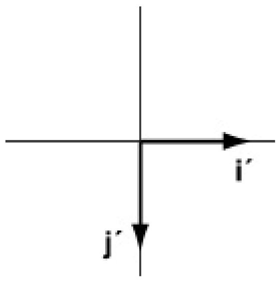
Rotation of 90° anticlockwise $A=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$



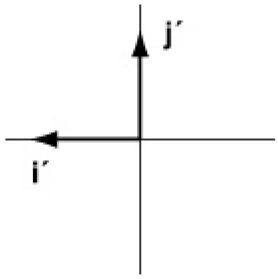
Rotation of 180° about (0,0) $B=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$



Reflection in x-axis $C=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$



Reflection in y-axis $\qquad D=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$



b

i $BC=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ (=D)

Reflection in y-axis

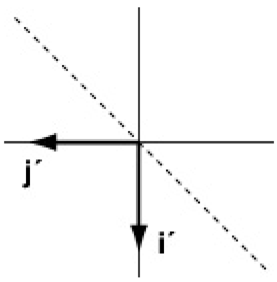
ii $CB=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ (=D)

Reflection in y-axis

iii $CD=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ (=B)

Rotation of 180° about (0,0)

iv $AD=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$



Reflection in line $y=-x$

v $BB=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Rotation of 360° about (0, 0) or Identity

vi

$$\begin{aligned}DAC &= \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\ &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\ &= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \quad (=A)\end{aligned}$$

Rotation of 90° anticlockwise about (0, 0)

vii

$$
\begin{aligned}
\text{CBD} &= \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\\
&= \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\\
&= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}
\end{aligned}
$$

Identity - no transformation

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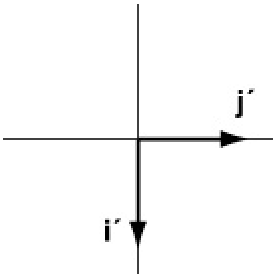
Matrix algebra
Exercise F, Question 3

Question:

Use a matrix product to find the single geometric transformation represented by a rotation of 270° anticlockwise about (0, 0) followed by a refection in the *x*-axis.

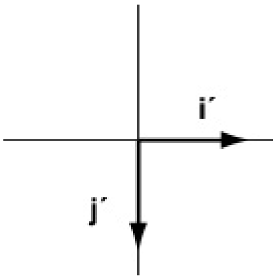
Solution:

Rotation of 270° about (0,0)



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}0\\-1\end{pmatrix}$$
$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix} \quad \therefore \text{ Matrix is } \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}.$$

Reflection is *x*-axis

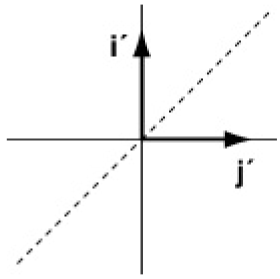


$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix}$$
$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\-1\end{pmatrix} \quad \therefore \text{ Matrix is } \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}.$$

Rotation of 270 followed by reflection in *x*-axis is:

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}0\\1\end{pmatrix}$$
$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix}$$



Reflection is $y = x$

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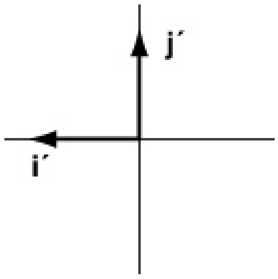
Matrix algebra
Exercise F, Question 4

Question:

Use matrices to show that a refection in the *y*-axis followed by a reflection in the line $y = -x$ is equivalent to a rotation of 90° anticlockwise about (0, 0).

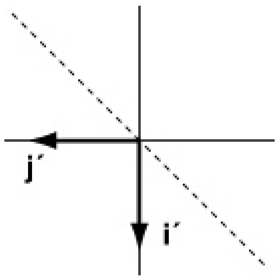
Solution:

Reflection in *y*-axis



$\therefore$ Matrix is $\mathrm{Y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

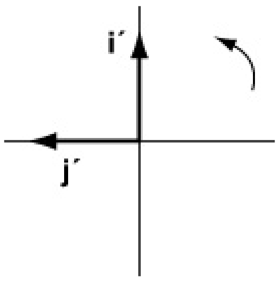
Reflection in $y = -x$



$\therefore$ Matrix $\mathrm{R} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$$\mathrm{RY} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$



i.e. Rotation of 90° anticlockwise about (0, 0).

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Matrix algebra
Exercise F, Question 5

Question:

The matrix **R** is given by $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

a Find $\mathbf{R}^2$.

b Describe the geometric transformation represented by $\mathbf{R}^2$.

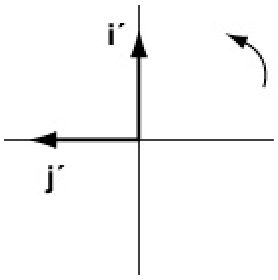
c Hence describe the geometric transformation represented by **R**.

d Write down $\mathbf{R}^8$.

Solution:

a $R^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b



i.e. R^2 represents rotation of 90° anticlockwise about (0, 0)

c R represents a rotation of 45° anticlockwise about (0, 0)

d R^8 will represent rotation of $8 \times 45^\circ = 360^\circ$

This is equivalent to no transformation

$\therefore \quad R^8 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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Matrix algebra
Exercise F, Question 6

Question:

P= $\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$, **Q**= $\begin{pmatrix} -1 & -2 \\ 3 & 5 \end{pmatrix}$

The transformation represented by the matrix **R** is the result of the transformation represented by the matrix **P** followed by the transformation represented by the matrix **Q**.

a Find **R**.

b Give a geometrical interpretation of the transformation represented by **R**.

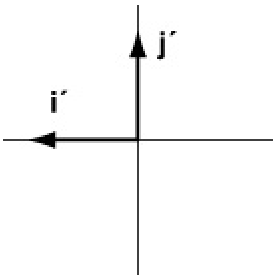
Solution:

a R= QP = $\begin{pmatrix} -1 & -2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

b

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



Reflection in y-axis

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Matrix algebra
Exercise F, Question 7

Question:

A$=\begin{pmatrix}5 & -7\\ 7 & -10\end{pmatrix}$, **B**$=\begin{pmatrix}4 & 3\\ 3 & 2\end{pmatrix}$, **C**$=\begin{pmatrix}-2 & 1\\ -1 & 1\end{pmatrix}$

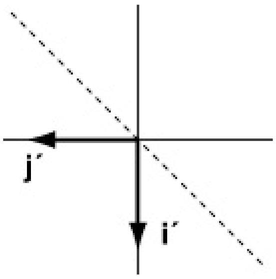
Matrices **A, B** and **C** represent three transformations. By combining the three transformations in the order **B**, followed by **A**, followed by **C** a single transformation is obtained.

Find a matrix representation of this transformation and interpret it geometrically.

Solution:

$$\begin{aligned}\text{CAB} &= \begin{pmatrix}-2 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}5 & -7\\ 7 & -10\end{pmatrix}\begin{pmatrix}4 & 3\\ 3 & 2\end{pmatrix}\\ &= \begin{pmatrix}-3 & 4\\ 2 & -3\end{pmatrix}\begin{pmatrix}4 & 3\\ 3 & 2\end{pmatrix}\\ &= \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\end{aligned}$$

$$\begin{pmatrix}1\\ 0\end{pmatrix} \to \begin{pmatrix}0\\ -1\end{pmatrix}$$
$$\begin{pmatrix}0\\ 1\end{pmatrix} \to \begin{pmatrix}-1\\ 0\end{pmatrix}$$



Reflection in the line $y = -x$

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Matrix algebra
Exercise F, Question 8

Question:

P= $\begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}$, **Q**= $\begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix}$, **R**= $\begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix}$

Matrices **P, Q** and **R** represent three transformations. By combining the three transformations in the order **R**, followed by **Q**, followed by **P** a single transformation is obtained.

Find a matrix representation of this transformation and interpret it geometrically.

Solution:

$$\begin{aligned} \text{PQR} &= \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \end{aligned}$$

Enlargement scale factor 8

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Matrix algebra
Exercise G, Question 1

Question:

Determine which of these matrices are singular and which are non-singular. For those that are non-singular find the inverse matrix.

a $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$

b $\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$

c $\begin{pmatrix} 2 & 5 \\ 0 & 0 \end{pmatrix}$

d $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$

e $\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$

f $\begin{pmatrix} 4 & 3 \\ 6 & 2 \end{pmatrix}$

Solution:

a

$$\begin{aligned} \det\begin{vmatrix} 3 & -1 \\ -4 & 2 \end{vmatrix} &= 6-(-4)\times(-1) \\ &= 6-4 \\ &= 2 \quad \neq 0 \end{aligned}$$

$\therefore$ the Matrix is non-singular

So inverse is $\frac{1}{2}\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$

or $\begin{pmatrix} 1 & 0.5 \\ 2 & 1.5 \end{pmatrix}$

b

$$\begin{aligned} \det\begin{vmatrix} 3 & 3 \\ -1 & -1 \end{vmatrix} &= -3-(-1)\times 3 \\ &= -3+3 \\ &= 0 \end{aligned}$$

$\therefore$ Matrix is singular.

c

$$\begin{aligned} \det\begin{vmatrix} 2 & 5 \\ 0 & 0 \end{vmatrix} &= 0-0 \\ &= 0 \end{aligned}$$

∴ Matrix is singular

d

$$\det\begin{vmatrix}1 & 2\\3 & 5\end{vmatrix} = 5-6$$
$$= -1 \quad \neq 0$$

∴ Matrix is non-singular

Inverse is $\frac{1}{-1}\begin{pmatrix}5 & -2\\-3 & 1\end{pmatrix} = \begin{pmatrix}-5 & 2\\3 & -1\end{pmatrix}$

e

$$\det\begin{vmatrix}6 & 3\\4 & 2\end{vmatrix} = 12-12$$
$$= 0$$

∴ Matrix is singular

f

$$\det\begin{vmatrix}4 & 3\\6 & 2\end{vmatrix} = 8-18$$
$$= -10 \quad \neq 0$$

∴ Matrix is non-singular

Inverse is $\frac{1}{-10}\begin{pmatrix}2 & -3\\-6 & 4\end{pmatrix}$
$$= \begin{pmatrix}-0.2 & 0.3\\0.6 & -0.4\end{pmatrix}$$

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Matrix algebra
Exercise G, Question 2

Question:

Find the value of a for which these matrices are singular.

a $\begin{pmatrix} a & 1+a \\ 3 & 2 \end{pmatrix}$

b $\begin{pmatrix} 1+a & 3-a \\ a+2 & 1-a \end{pmatrix}$

c $\begin{pmatrix} 2+a & 1-a \\ 1-a & a \end{pmatrix}$

Solution:

a

$$\begin{aligned} \det\begin{vmatrix} a & 1+a \\ 3 & 2 \end{vmatrix} &= 2a-3(1+a) \\ &= 2a-3-3a \\ &= -3-a \end{aligned}$$

Matrix is singular for $a=-3$

b

$$\begin{aligned} \text{Let A} &= \begin{pmatrix} 1+a & 3-a \\ a+2 & 1-a \end{pmatrix} \\ \det \text{A} &= (1+a)(1-a)-(3-a)(a+2) \\ &= 1-a^2-(-a^2+a+6) \\ &= 1-a^2+a^2-a-6 \\ &= -a-5 \\ \det \text{A} &= 0 \quad \Rightarrow \quad a=-5 \end{aligned}$$

c

$$\begin{aligned} \text{Let B} &= \begin{pmatrix} 2+a & 1-a \\ 1-a & a \end{pmatrix} \\ \det \text{B} &= 2a+a^2-(1-a)^2 \\ &= 2a+a^2-1+2a-a^2 \\ &= 4a-1 \\ \det \text{B} &= 0 \quad \Rightarrow \quad a=\frac{1}{4} \end{aligned}$$

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Matrix algebra
Exercise G, Question 3

Question:

Find inverses of these matrices.

a $\begin{pmatrix} a & 1+a \\ 1+a & 2+a \end{pmatrix}$

b $\begin{pmatrix} 2a & 3b \\ -a & -b \end{pmatrix}$

Solution:

a

$$\begin{aligned}
\text{Let A} &= \begin{pmatrix} a & 1+a \\ 1+a & 2+a \end{pmatrix} \\
\det \text{A} &= 2a + a^2 - (1+a)^2 \\
&= 2a + a^2 - 1 - 2a - a^2 \\
&= -1 \\
\text{A}^{-1} &= \frac{1}{-1}\begin{pmatrix} 2+a & -(1+a) \\ -(1+a) & a \end{pmatrix} = \begin{pmatrix} -[2+a] & (1+a) \\ (1+a) & -a \end{pmatrix}
\end{aligned}$$

b

$$\begin{aligned}
\text{Let B} &= \begin{pmatrix} 2a & 3b \\ -a & -b \end{pmatrix} \\
\det \text{B} &= -2ab - (-a) \times 3b \\
&= -2ab + 3ab \\
&= ab \\
\text{B}^{-1} &= \frac{1}{ab}\begin{pmatrix} -b & -3b \\ a & 2a \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{a} & -\frac{3}{a} \\ \frac{1}{b} & \frac{2}{b} \end{pmatrix} \quad \text{provided that } ab \neq 0
\end{aligned}$$

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Matrix algebra
Exercise G, Question 4

Question:

a Given that **ABC** =**I**, prove that $\mathbf{B}^{-1} = \mathbf{CA}$.

b Given that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & -6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}$, find **B**.

Solution:

a

$$\begin{aligned} & ABC &&= I \\ \Rightarrow\ & A^{-1}ABC &&= A^{-1}I \\ \Rightarrow\ & BC &&= A^{-1} \\ \Rightarrow\ & BCC^{-1} &&= A^{-1}C^{-1} \\ \Rightarrow\ & B &&= A^{-1}C^{-1} = (CA)^{-1} \\ \therefore\ & B^{-1} &&= CA \end{aligned}$$

b

$$\begin{aligned} CA &= \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -6 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix} \\ \therefore\ (CA)^{-1} &= \frac{1}{-3+4}\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix} \\ \therefore\ B &= \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix} \end{aligned}$$

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Matrix algebra
Exercise G, Question 5

Question:

a Given that **AB** =**C**, find an expression for **B**.

b Given further that **A**= $\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$ and **C**= $\begin{pmatrix} 3 & 6 \\ 1 & 22 \end{pmatrix}$, find **B**.

Solution:

a

$$\begin{aligned} AB &= C \\ \Rightarrow \quad A^{-1}AB &= A^{-1}C \\ \Rightarrow \quad B &= A^{-1}C \end{aligned}$$

b

$$\begin{aligned} A &= \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix} \quad \Rightarrow \text{det A} = 6 - -4 = 10 \\ \therefore \quad A^{-1} &= \frac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix} \\ \therefore \quad B &= A^{-1}C \\ &= \frac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 1 & 22 \end{pmatrix} \\ &= \frac{1}{10}\begin{pmatrix} 10 & 40 \\ -10 & 20 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix} \end{aligned}$$

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Matrix algebra
Exercise G, Question 6

Question:

a Given that **BAC** =**B**, where **B** is a non-singular matrix, find an expression for **A**.

b When **C**= $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$, find **A**.

Solution:

a

$$
\begin{aligned}
& \mathrm{BAC} = \mathrm{B} \\
\Rightarrow\ & \mathrm{B}^{-1}\mathrm{BAC} = \mathrm{B}^{-1}\mathrm{B} \\
\Rightarrow\ & \mathrm{AC} = \mathrm{I} \\
\Rightarrow\ & \mathrm{A} = \mathrm{C}^{-1}
\end{aligned}
$$

b

$$
\begin{aligned}
& \mathrm{C} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \\
& \det \mathrm{C} = 10 - 9 = 1 \\
\therefore\ & \mathrm{C}^{-1} = \frac{1}{1}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \\
\therefore\ & \mathrm{A} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}
\end{aligned}
$$

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Matrix algebra
Exercise G, Question 7

Question:

The matrix $\mathbf{A}=\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$ and $\mathbf{AB}=\begin{pmatrix} 4 & 7 & -8 \\ -8 & -13 & 18 \end{pmatrix}$. Find the matrix $\mathbf{B}$.

Solution:

$$A = \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} \quad \Rightarrow \quad \det A = 6 - (-4) \times (-1) = 2$$

$$\therefore \quad A^{-1} = \frac{1}{2}\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 4 & 7 & -8 \\ -8 & -13 & 18 \end{pmatrix} \quad (\times \text{ on left by} A^{-1})$$

$$\Rightarrow \quad B = \frac{1}{2}\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 4 & 7 & -8 \\ -8 & -13 & 18 \end{pmatrix}$$

$$B = \frac{1}{2}\begin{pmatrix} 4 & 8 & -6 \\ 0 & 2 & 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 4 & -3 \\ 0 & 1 & 2 \end{pmatrix}$$

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Matrix algebra
Exercise G, Question 8

Question:

The matrix $\mathbf{B} = \begin{pmatrix} 5 & -4 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 11 & -1 \\ -8 & 9 \\ -2 & -1 \end{pmatrix}$. Find the matrix **A**.

Solution:

$$\mathrm{B} = \begin{pmatrix} 5 & -4 \\ 2 & 1 \end{pmatrix} \quad \Rightarrow \det \mathrm{B} = 5 + 8 = 13$$

$$\mathrm{B}^{-1} = \frac{1}{13}\begin{pmatrix} 1 & 4 \\ -2 & 5 \end{pmatrix}$$

$$\mathrm{AB} = \begin{pmatrix} 11 & -1 \\ -8 & 9 \\ -2 & -1 \end{pmatrix} \quad (\times \text{ on right by } \mathrm{B}^{-1})$$

$$\Rightarrow \quad \mathrm{ABB}^{-1} = \begin{pmatrix} 11 & -1 \\ -8 & 9 \\ -2 & -1 \end{pmatrix}\mathrm{B}^{-1}$$

$$\therefore \quad \mathrm{A} = \frac{1}{13}\begin{pmatrix} 11 & -1 \\ -8 & 9 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -2 & 5 \end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix} 13 & 39 \\ -26 & 13 \\ 0 & -13 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & -1 \end{pmatrix}$$

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Matrix algebra
Exercise G, Question 9

Question:

The matrix **A**= $\begin{pmatrix} 3a & b \\ 4a & 2b \end{pmatrix}$, where a and b are non-zero constants.

a Find $\mathbf{A}^{-1}$.

The matrix **B**= $\begin{pmatrix} -a & b \\ 3a & 2b \end{pmatrix}$ and the matrix **X** is given by **B**= **XA**.

b Find **X**.

Solution:

a

$$\text{A} = \begin{pmatrix} 3a & b \\ 4a & 2b \end{pmatrix} \quad \Rightarrow \quad \det \text{A} = 6ab - 4ab = 2ab$$

$$\therefore \quad \text{A}^{-1} = \frac{1}{2ab}\begin{pmatrix} 2b & -b \\ -4a & 3a \end{pmatrix}$$

b

$$\begin{aligned}
\text{B} &= \text{XA} \\
\Rightarrow \quad \text{BA}^{-1} &= \text{XAA}^{-1} \\
\therefore \quad \text{X} &= \text{BA}^{-1} \\
\text{So} \quad \text{X} &= \begin{pmatrix} -a & b \\ 3a & 2b \end{pmatrix}\begin{pmatrix} 2b & -b \\ -4a & 3a \end{pmatrix} \times \frac{1}{2ab} \\
&= \frac{1}{2ab}\begin{pmatrix} -6ab & 4ab \\ -2ab & 3ab \end{pmatrix} \\
\therefore \quad \text{X} &= \begin{pmatrix} -3 & 2 \\ -1 & 3/2 \end{pmatrix}
\end{aligned}$$

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Matrix algebra
Exercise G, Question 10

Question:

The matrix $\mathbf{A}=\begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix}$ and the matrix $\mathbf{B}=\begin{pmatrix} 2b & -2a \\ -b & a \end{pmatrix}$.

a Find det ($\mathbf{A}$) and det ($\mathbf{B}$).

b Find $\mathbf{AB}$.

Solution:

a

$$\mathrm{A} = \begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix} \quad \Rightarrow \quad \det \mathrm{A} = 2ab - 2ab = 0$$

$$\mathrm{B} = \begin{pmatrix} 2b & -2a \\ -b & a \end{pmatrix} \quad \Rightarrow \quad \det \mathrm{B} = 2ab - 2ab = 0$$

b

$$\begin{aligned} \mathrm{AB} &= \begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix}\begin{pmatrix} 2b & -2a \\ -b & a \end{pmatrix} \\ &= \begin{pmatrix} 2ab - 2ab & -2a^2 + 2a^2 \\ 2b^2 - 2b^2 & -2ab + 2ab \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

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Matrix algebra
Exercise G, Question 11

Question:

The non-singular matrices **A** and **B** are commutative (i.e. **AB** = **BA**) and **ABA** =**B**.

a Prove that $\mathbf{A}^2 = \mathbf{I}$.

Given that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, by considering a matrix **B** of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

b show that $a = d$ and $b = c$.

Solution:

a

Given $\quad AB = BA$
and $\quad ABA = B$

$\Rightarrow \quad A(AB) = B$

$\Rightarrow \quad A^2B = B$

$\Rightarrow \quad A^2BB^{-1} = BB^{-1}$

$\Rightarrow \quad A^2 = I$

b

$$\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

$\mathbf{AB} = \mathbf{BA} \Rightarrow \quad b = c$
$\qquad\qquad\qquad\quad d = a$

i.e. $a = d$ and $b = c$

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Matrix algebra
Exercise H, Question 1

Question:

The matrix $\mathbf{R}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

a Give a geometrical interpretation of the transformation represented by **R**.

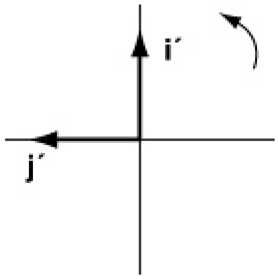
b Find $\mathbf{R}^{-1}$.

c Give a geometrical interpretation of the transformation represented by $\mathbf{R}^{-1}$.

Solution:

a

$$\begin{aligned}(1,0) &\rightarrow (0,1)\\ (0,1) &\rightarrow (-1,0)\end{aligned}$$



R represents a rotation of 90° anticlockwise about (0, 0)

b

$$\begin{aligned}\det \mathbf{R} &= 0 - -1 = 1\\ \therefore \quad \mathbf{R}^{-1} &= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\end{aligned}$$

c $\mathbf{R}^{-1}$represents a rotation of −90° anticlockwise about (0,0)

(or … 90° clockwise … or … 270° anticlockwise …)

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Matrix algebra
Exercise H, Question 2

Question:

a The matrix $\mathbf{S}=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$

i Give a geometrical interpretation of the transformation represented by **S**.

ii Show that $\mathbf{S}^2=\mathbf{I}$.

iii Give a geometrical interpretation of the transformation represented by $\mathbf{S}^{-1}$.

b The matrix $\mathbf{T}=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$

i Give a geometrical interpretation of the transformation represented by **T**.

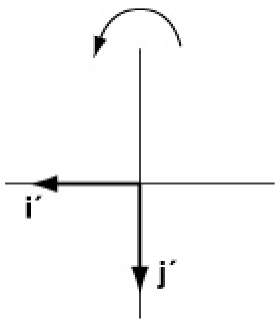
ii Show that $\mathbf{T}^2=\mathbf{I}$.

iii Give a geometrical interpretation of the transformation represented by $\mathbf{T}^{-1}$.

c Calculate det(**S**) and det(**T**) and comment on their values in the light of the transformations they represent.

Solution:

a i $\begin{matrix}(1,0) & \to & (-1,0)\\ (0,1) & \to & (0,-1)\end{matrix}$



S represents a rotation of 180° about (0,0)

ii $\mathbf{S}^2$ will be a rotation of $180+180=360^\circ$ about (0,0) $\therefore\ \mathbf{S}^2=\mathbf{I}$

or $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\mathbf{I}$

iii $\mathbf{S}^{-1}=\mathbf{S}=$ rotation of 180° about (0,0)

b i

$(1,0) \to (0,-1)$
$(0,1) \to (-1,0)$

T represents a reflection in the line $y=-x$

ii $\mathbf{T}^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

iii $\mathbf{T}^{-1} = \mathbf{T}$ = reflection in the line $y = -x$

c

$\det \mathbf{S} = 1 - 0 = 1$
$\det \mathbf{T} = 0 - 1 = -1$

For both **S** and **T**, area is unaltered

T represents a reflection and $\therefore$ has a negative determinant. Orientation is reversed

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Matrix algebra
Exercise H, Question 3

Question:

The matrix **A** represents a reflection in the line $y = x$ and the matrix **B** represents a rotation of 270° about (0, 0).

a Find the matrix **C**= **BA** and interpret it geometrically.

b Find $\mathbf{C}^{-1}$ and give a geometrical interpretation of the transformation represented by $\mathbf{C}^{-1}$.

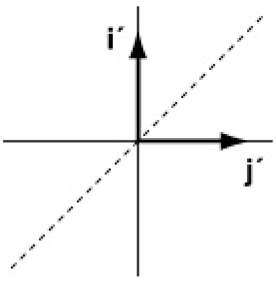
c Find the matrix **D**= **AB** and interpret it geometrically.

d Find $\mathbf{D}^{-1}$ and give a geometrical interpretation of the transformation represented by $\mathbf{D}^{-1}$.

Solution:

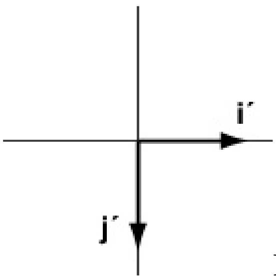
a

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



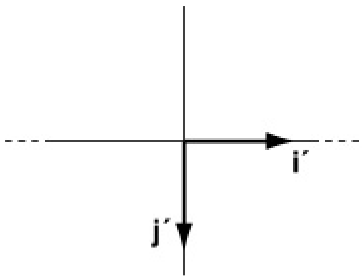
Reflection in $y = x$

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$



Rotation of 270° (about (0,0))

$$\mathbf{C} = \mathbf{BA} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

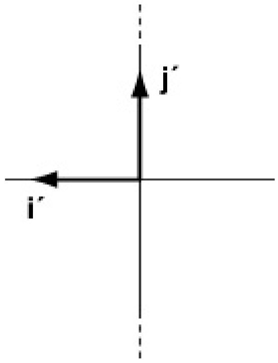


C represents a reflection in the line $y = 0$ (or the x-axis)

b $\mathbf{C}^{-1} = \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a reflection in the line $y = 0$

c

$$\mathbf{D} = \mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



D represents a reflection in the line $x = 0$ (or the y-axis)

d $\mathbf{D}^{-1} = \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is a reflection in the line $x = 0$

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Matrix algebra
Exercise I, Question 1

Question:

The matrix **A**= $\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$ is used to transform the rectangle *R* with vertices at the points (0, 0), (0, 1), (4, 1) and (4, 0).

a Find the coordinates of the vertices of the image of *R*.

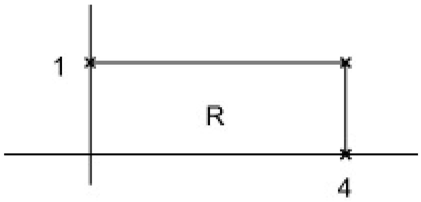
b Calculate the area of the image of *R*.

Solution:

a $\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 4 & 4 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 7 & 8 \\ 0 & 3 & 19 & 16 \end{pmatrix}$

Coordinates of image are: (0,0); (–1,3); (7,19); (8,16)

b



Area of R= $4 \times 1 = 4$

$\det \mathbf{A} = 6 - -4 = 10$

$\therefore$ Area of image $= 10 \times 4$
$= 40$.

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Matrix algebra
Exercise I, Question 2

Question:

The triangle T has vertices at the points $(-3.5,\ 2.5)$, $(-16,\ 10)$ and $(-7,\ 4)$.

a Find the coordinates of the vertices of T under the transformation given by the matrix $\mathbf{M} = \begin{pmatrix} -1 & -1 \\ 3 & 5 \end{pmatrix}$.

b Show that the area of the image of T is 7.5.

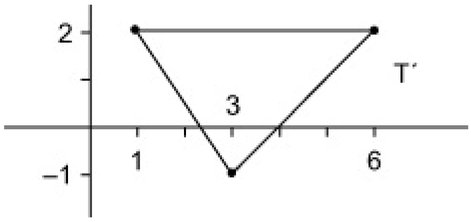
c Hence find the area of T.

Solution:

a $\begin{pmatrix} -1 & -1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -3.5 & -16 & -7 \\ 2.5 & 10 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 6 & 3 \\ 2 & 2 & -1 \end{pmatrix}$

Coordinates of T′ are $(1,2)$; $(6,2)$; $(3,-1)$

b



Area of $\mathrm{T}' = \frac{1}{2} \times 5 \times 3 = 7.5$

c

$$\begin{aligned} \det \mathbf{M} = -5 + 3 &= -2 \\ \therefore \text{ Area of T} \times |-2| &= \text{Area of T}' \\ \Rightarrow \quad \text{Area of T} &= \frac{7.5}{2} \\ &= 3.75 \end{aligned}$$

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Matrix algebra
Exercise I, Question 3

Question:

The rectangle R has vertices at the points $(-1,\ 0)$, $(0\ ,\ -3)$, (4, 0) and (3, 3).

The matrix $\mathbf{A}=\begin{pmatrix}-2 & 3-a\\ 1 & a\end{pmatrix}$, where a is a constant.

a Find, in terms of a, the coordinates of the vertices of the image of R under the transformation given by **A**.

b Find det(**A**), leaving your answer in terms of a.

Given that the area of the image of R is 75

c find the positive value of a.

Solution:

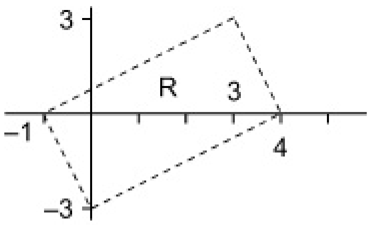
a $\begin{pmatrix}-2 & 3-a\\ 1 & a\end{pmatrix}\begin{pmatrix}-1 & 0 & 4 & 3\\ 0 & -3 & 0 & 3\end{pmatrix}=\begin{pmatrix}+2 & 3a-9 & -8 & 3-3a\\ -1 & -3a & 4 & 3+3a\end{pmatrix}$

Image of R is : $(+2\ ,-1);\ (3a-9\ ,-3a);\ (-8,4);\ (3-3a,3+3a)$

b

$$\begin{aligned}\det\ \mathbf{A}\ &= -2a-3+a\\ &= -a-3\end{aligned}$$

$$\begin{aligned}\text{Area of R}\ &= \left(\frac{1}{2}\times 5\times 3\right)\times 2.\\ &= 15\end{aligned}$$



c

$$\begin{aligned}\text{Area of R}\times|\det\ \mathbf{A}|\ &= 75\\ \therefore\quad |\det\ \mathbf{A}|\ &= \frac{75}{15}=5\\ \text{So}\quad |-a-3|\ &= 5\\ \Rightarrow\quad -a-3\ &= 5\ \text{ or }\ a+3=5\end{aligned}$$

$\therefore$ positive value of $a=2$

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Matrix algebra
Exercise I, Question 4

Question:

P$=\begin{pmatrix}2 & -4\\3 & 1\end{pmatrix}$, **Q**$=\begin{pmatrix}1 & 2\\-1 & 4\end{pmatrix}$, **R**$=\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}$.

A rectangle of area 5 cm^2 is transformed by the matrix **X**. Find the area of the image of the rectangle when **X** is:

a P

b Q

c R

d RQ

e QR

f RP

Solution:

a $\det \mathbf{P} = 2 + 12 = 14 \quad \therefore$ area of image is 70 cm^2

b $\det \mathbf{Q} = 4 + 2 = 6 \quad \therefore$ area of image is 30 cm^2

c $\det \mathbf{R} = 1 - 4 = -3 \quad \therefore$ area of image is 15 cm^2

d $\det \mathbf{RQ} = \det \mathbf{R} \times \det \mathbf{Q} = -18 \; \therefore$ area of image is 90 cm^2

e $\det \mathbf{QR} = \det \mathbf{Q} \times \det \mathbf{R} = -18 \; \therefore$ area of image is 90 cm^2

f $\det \mathbf{RP} = \det \mathbf{R} \times \det \mathbf{P} = -42 \; \therefore$ area of image is 210 cm^2

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Matrix algebra
Exercise I, Question 5

Question:

The triangle T has area 6 cm^2 and is transformed by the matrix $\begin{pmatrix} a & 3 \\ 3 & a+2 \end{pmatrix}$, where a is a constant, into triangle T'.

a Find det(**A**) in terms of a.

Given that the area of T' is 36 cm^2

b find the possible values of a.

Solution:

a

$$\begin{aligned} \det \mathbf{A} &= a(a+2)-9 \\ &= a^2+2a-9 \end{aligned}$$

b

$$\begin{aligned} \text{Area of T} \times |\det \mathbf{A}| &= \text{Area of T}' \\ \therefore\ 6 \times |\det \mathbf{A}| &= 36 \\ \therefore\ \det \mathbf{A} &= \pm 6 \\ \Rightarrow a^2+2a-9 &= 6 \\ a^2+2a-15 &= 0 \\ (a+5)(a-3) &= 0 \\ \therefore\ a = 3 \text{ or } -5 \end{aligned}$$

or

$$\begin{aligned} \Rightarrow a^2+2a-9 &= -6 \\ a^2+2a-3 &= 0 \\ (a+3)(a-1) &= 0 \\ a &= 1 \text{ or } -3 \end{aligned}$$

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Matrix algebra
Exercise J, Question 1

Question:

Use inverse matrices to solve the following simultaneous equations

a $7x + 3y = 6$

$-5x - 2y = -5$

b $4x - y = -1$

$-2x + 3y = 8$

Solution:

a $\begin{pmatrix} 7 & 3 \\ -5 & -2 \end{pmatrix} = \mathbf{A} \quad \Rightarrow \quad \det \mathbf{A} = -14 + 15 = 1$

$\therefore \mathbf{A}^{-1} = \frac{1}{1}\begin{pmatrix} -2 & -3 \\ 5 & 7 \end{pmatrix}$

$\therefore \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 6 \\ -5 \end{pmatrix}$

$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 6 \\ -5 \end{pmatrix}$

$= \begin{pmatrix} -12 + 15 \\ 30 - 35 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

$\therefore \quad x = 3, \; y = -5$

b $\mathbf{B} = \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix} \quad \Rightarrow \quad \det \mathbf{B} = 12 - (-2)(-1) = 10$

$\therefore \quad \mathbf{B}^{-1} = \frac{1}{10}\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$

$\therefore \quad \mathbf{B}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \end{pmatrix} \quad \Rightarrow \quad \mathbf{B}^{-1}\begin{pmatrix} -1 \\ 8 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

So $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 8 \end{pmatrix}$

$= \frac{1}{10}\begin{pmatrix} -3 + 8 \\ -2 + 32 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 3 \end{pmatrix}$

$\therefore \quad x = 0.5, \; y = 3$

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Matrix algebra
Exercise J, Question 2

Question:

Use inverse matrices to solve the following simultaneous equations

a $4x - y = 11$

$3x + 2y = 0$

b $5x + 2y = 3$

$3x + 4y = 13$

Solution:

a $\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} \Rightarrow \det \mathbf{A} = 8 + 3 = 11$

$\therefore \mathbf{A}^{-1} = \frac{1}{11}\begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix}$

So $\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 11 \\ 0 \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 11 \\ 0 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 22 \\ -33 \end{pmatrix}$

$\therefore \quad x = 2, \; y = -3$

b $\mathbf{B} = \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow \det \mathbf{B} = 20 - 6 = 14$

$\therefore \mathbf{B}^{-1} = \frac{1}{14}\begin{pmatrix} 4 & -2 \\ -3 & 5 \end{pmatrix}$

So $\mathbf{B}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 13 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{B}^{-1}\begin{pmatrix} 3 \\ 13 \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 4 & -2 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 13 \end{pmatrix}$

$= \frac{1}{14}\begin{pmatrix} 12 - 26 \\ -9 + 65 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} -14 \\ 56 \end{pmatrix}$

$\therefore \quad x = -1, \; y = 4$

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Matrix algebra
Exercise K, Question 1

Question:

The matrix $\mathbf{A}=\begin{pmatrix}3 & 1\\4 & 2\end{pmatrix}$ transforms the triangle PQR into the triangle with coordinates $(6\,,\,-2)$, (4, 4), (0, 8).

Find the coordinates of P, Q and R.

Solution:

$$\mathbf{A}=\begin{pmatrix}3 & 1\\4 & 2\end{pmatrix} \Rightarrow \text{det } \mathbf{A}=6-4=2.$$

$$\therefore \quad \mathbf{A}^{-1}=\frac{1}{2}\begin{pmatrix}2 & -1\\-4 & 3\end{pmatrix}$$

$$\mathbf{A}(\Delta PQR)=\begin{pmatrix}6 & 4 & 0\\-2 & 4 & 8\end{pmatrix}$$

$$\therefore \Delta PQR \text{ given by } \frac{1}{2}\begin{pmatrix}2 & -1\\-4 & 3\end{pmatrix}\begin{pmatrix}6 & 4 & 0\\-2 & 4 & 8\end{pmatrix}$$
$$=\frac{1}{2}\begin{pmatrix}14 & 4 & -8\\-30 & -4 & 24\end{pmatrix}$$
$$=\begin{pmatrix}7 & 2 & -4\\-15 & -2 & 12\end{pmatrix}$$

$\therefore P$ is $(7\,,-15)$, Q is $(2\,,-2)$, R is $(-4,\ 12)$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Matrix algebra
Exercise K, Question 2

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 4 & 1 & 9 \\ 1 & 9 & 4 \end{pmatrix}$.

Find the matrix **B**.

Solution:

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} \Rightarrow \det \mathbf{A} = 1 + 6 = 7$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1}(\mathbf{AB}) = \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 & 9 \\ 1 & 9 & 4 \end{pmatrix}$$

$$\therefore \quad \mathbf{B} = \frac{1}{7}\begin{pmatrix} 7 & 28 & 21 \\ -7 & 7 & -14 \end{pmatrix}$$

$$\therefore \quad \mathbf{B} = \begin{pmatrix} 1 & 4 & 3 \\ -1 & 1 & -2 \end{pmatrix}$$

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Matrix algebra
Exercise K, Question 3

Question:

A= $\begin{pmatrix} -2 & 1 \\ 7 & -3 \end{pmatrix}$, **B**= $\begin{pmatrix} 4 & 1 \\ -5 & -1 \end{pmatrix}$, **C**= $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$.

The matrices **A**, **B** and **C** represent three transformations. By combining the three transformations in the order **A**, followed by **B**, followed by **C**, a simple single transformation is obtained which is represented by the matrix **R**.

a Find **R**.

b Give a geometrical interpretation of the transformation represented by **R**.

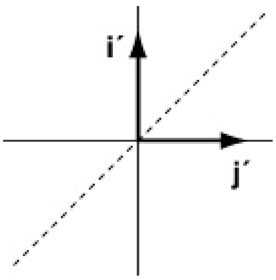
c Write down the matrix $\mathbf{R}^2$.

Solution:

a

$$\begin{aligned}\mathbf{R} &= \mathbf{C\,B\,A} \\ &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -5 & -1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 7 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}\end{aligned}$$

$$\mathbf{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



b **R** represents a reflection in the line $y = x$

c $\mathbf{R}^2 = \mathrm{I}$

Since repeating a reflection twice returns an object to its original position.

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Matrix algebra
Exercise K, Question 4

Question:

The matrix **Y** represents a rotation of 90° about (0, 0).

a Find **Y**.

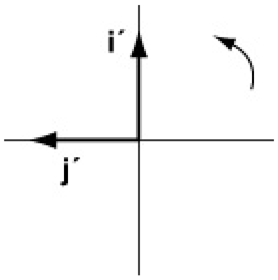
The matrices **A** and **B** are such that **AB** =**Y**. Given that **B**= $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$

b find **A**.

c Simplify **ABABABAB**.

Solution:

a



$$\mathbf{Y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

b

$$\mathbf{AB} = \mathbf{Y} \quad \Rightarrow \quad \mathbf{A} = \mathbf{YB}^{-1}.$$

$$\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow \quad \det \mathbf{B} = 3 - 4 = -1$$

$$\therefore \quad \mathbf{B}^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$\therefore \quad \mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$$

c $\mathbf{ABABABAB} = \mathbf{Y}^4$

$= \text{rotation of } 4 \times 90 = 360^\circ \text{ about } (0,\ 0)$
$= \mathrm{I}$

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Matrix algebra
Exercise K, Question 5

Question:

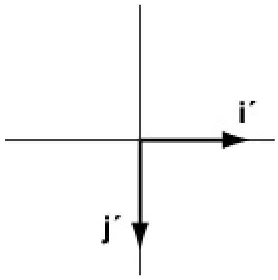
The matrix **R** represents a reflection in the *x*-axis and the matrix **E** represents an enlargement of scale factor 2 centre (0, 0).

a Find the matrix **C**= **ER** and interpret it geometrically.

b Find $\mathbf{C}^{-1}$ and give a geometrical interpretation of the transformation represented by $\mathbf{C}^{-1}$.

Solution:

Reflection in *x*-axis



$$\Rightarrow \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Enlargement S.F. 2 centre (0, 0)

$$\Rightarrow \mathrm{E} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

a

$$\mathbf{C} = \mathbf{ER} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

Reflection in the *x*-axis and enlargement SF 2. Centre (0, 0)

b $$\mathbf{C}^{-1} = \frac{1}{-4}\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Reflection in the *x*-axis and enlargement scale factor $\frac{1}{2}$. Centre (0, 0)

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Matrix algebra
Exercise K, Question 6

Question:

The quadrilateral R of area 4cm^2 is transformed to R' by the matrix $\mathbf{P}=\begin{pmatrix} 1+p & p \\ 2-p & p \end{pmatrix}$, where p is a constant.

a Find det(**P**) in terms of p.

Given that the area of $R' = 12\text{cm}^2$

b find the possible values of p.

Solution:

a

$$\mathbf{P}=\begin{pmatrix} 1+p & p \\ 2-p & p \end{pmatrix} \Rightarrow \quad \det \mathbf{P} = p(1+p)-p(2-p)$$
$$= p+p^2-2p+p^2$$
$$= 2p^2-p.$$

b

$$\begin{aligned}
\text{Area of R} \times |\det p| &= \text{Area of } R^1 \\
\therefore \quad 4\times|\det p| &= 12 \\
\therefore \quad \det p &= \pm 3 \\
\text{So} \quad 2p^2-p &= 3 \\
\Rightarrow \quad 2p^2-p-3 &= 0 \\
(2p-3)(p+1) &= 0
\end{aligned}$$

$$p=-1 \text{ or } \frac{3}{2}$$

$$\begin{aligned}
\text{or} \quad 2p^2-p &= -3 \\
\Rightarrow \quad 2p^2-p+3 &= 0
\end{aligned}$$

$$\text{Discrimininat is } (-1)^2-4\times 3\times 2=-23$$
$$<0$$
$$\therefore \text{ no solutions}$$
$$\text{so} \quad p=-1 \text{ or } \frac{3}{2} \text{ are the only solutions}$$

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Matrix algebra
Exercise K, Question 7

Question:

The matrix $\mathbf{A}=\begin{pmatrix} a & b \\ 2a & 3b \end{pmatrix}$, where a and b are non-zero constants.

a Find $\mathbf{A}^{-1}$.

The matrix $\mathbf{Y}=\begin{pmatrix} a & 2b \\ 2a & b \end{pmatrix}$ and the matrix $\mathbf{X}$ is given by $\mathbf{XA}=\mathbf{Y}$.

b Find $\mathbf{X}$.

Solution:

a

$$\mathbf{A} = \begin{pmatrix} a & b \\ 2a & 3b \end{pmatrix} \quad \Rightarrow \quad \det \mathbf{A} = 3ab - 2ab = ab$$

$$\therefore \mathbf{A}^{-1} = \frac{1}{ab}\begin{pmatrix} 3b & -b \\ -2a & a \end{pmatrix} = \begin{pmatrix} \frac{3}{a} & -\frac{1}{a} \\ \frac{-2}{b} & \frac{1}{b} \end{pmatrix}$$

b

$$\mathbf{XA} = \mathbf{Y} \quad \Rightarrow \quad \mathbf{X} = \mathbf{YA}^{-1}$$

$$\therefore \quad \mathbf{X} = \begin{pmatrix} a & 2b \\ 2a & b \end{pmatrix}\begin{pmatrix} \frac{3}{a} & -\frac{1}{a} \\ -\frac{2}{b} & \frac{1}{b} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix}$$

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Matrix algebra
Exercise K, Question 8

Question:

The 2×2, non-singular matrices, **A**, **B** and **X** satisfy $\mathbf{XB} = \mathbf{BA}$.

a Find an expression for **X**.

b Given that $\mathbf{A}=\begin{pmatrix} 5 & 3 \\ 0 & -2 \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$, find **X**.

Solution:

a

$$\begin{aligned} \mathbf{XB} &= \mathbf{BA} \\ \therefore (\mathbf{XB})\mathbf{B}^{-1} &= \mathbf{BAB}^{-1} \end{aligned}$$

$$\text{i.e.}\quad \mathbf{X} = \mathbf{BAB}^{-1} \qquad (\because \mathbf{BB}^{-1} = \mathrm{I})$$

b

$$\begin{aligned} \mathbf{B} &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \Rightarrow \ \det \mathbf{B} = -2-(-1) = -1 \\ \therefore \quad \mathbf{B}^{-1} &= \frac{1}{-1}\begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \\ \therefore \quad \mathbf{X} &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix} \\ \mathbf{X} &= \begin{pmatrix} 6 & 2 \\ -4 & -3 \end{pmatrix} \end{aligned}$$

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Series
Exercise A, Question 1

Question:

Write out each of the following as a sum of terms, and hence calculate the sum of the series.

a $\sum_{r=1}^{10} r$

b $\sum_{p=3}^{8} p^2$

c $\sum_{r=1}^{10} r^3$

d $\sum_{p=1}^{10} (2p^2 + 3)$

e $\sum_{r=0}^{5} (7r + 1)^2$

f $\sum_{i=1}^{4} 2i(3 - 4i^2)$

Solution:

a $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$

b $3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 = 199$

c $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3 = 3025$

{notice that this result is the square of the result for (a)}

d $5 + 11 + 21 + 35 + 53 + 75 + 101 + 131 + 165 + 203 = 800$

e $1 + 64 + 225 + 484 + 841 + 1296 = 2911$

f $-2 - 52 - 198 - 488 = -740$

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Edexcel AS and A Level Modular Mathematics

Series
Exercise A, Question 2

Question:

Write each of the following as a sum of terms, showing the first three terms and the last term.

a $\sum_{r=1}^{n}(7r-1)$

b $\sum_{r=1}^{n}(2r^3+1)$

c $\sum_{j=1}^{n}(j-4)(j+4)$

d $\sum_{p=3}^{k} p(p+3)$

Solution:

a $6+13+20+\ldots+(7n-1)$

b $3+17+55+\ldots+(2n^3+1)$

c $-15-12-7+\ldots+(n-4)(n+4)$

d $18+28+40+\ldots+k(k+3)$

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Series
Exercise A, Question 3

Question:

In each part of this question write out, as a sum of terms, the two series defined by $\sum f(r)$; for example, in part **c**, write out the series $\sum_{r=1}^{10} r^2$ and $\sum_{r=1}^{10} r$. Hence, state whether the given statements relating their sums are true or not.

a $\sum_{r=1}^{n}(3r+1)=\sum_{r=2}^{n+1}(3r-2)$

b $\sum_{r=1}^{n}2r=\sum_{r=0}^{n}2r$

c $\sum_{r=1}^{10}r^2=\left(\sum_{1}^{10}r\right)^2$

d $\sum_{r=1}^{4}r^3=\left(\sum_{r=1}^{4}r\right)^2$

e $\sum_{r=1}^{n}\left(3r^2+4\right)=3\sum_{r=1}^{n}r^2+4$

Solution:

a The two series are exactly the same, $4+7+10+\ldots+(3n+1)$, and so their sums are the same.

b The two series are exactly the same, $2+4+6+\ldots+2n$, and so their sums are the same.

c The statement is not true.

$$\sum_{r=1}^{r=10}r^2=1^2+2^2+3^2+\ldots+10^2=385 \text{ (using your calculator)}$$

$$\left(\sum_{r=1}^{10}r\right)^2=(1+2+3+\ldots 10)^2=55^2=3025.$$

[This one example is enough to prove $\sum_{r=1}^{n}r^2=\left(\sum_{r=1}^{n}r\right)^2$ for all n is not true]

d This statement is true.

$$\sum_{r=1}^{4}r^3 = 1^3+2^3+3^3+4^3=100$$

$$\left(\sum_{r=1}^{4}r\right)^2 = (1+2+3+4)^2=10^2=100$$

[This does not prove that $\sum_{r=1}^{n} r^3 = \left(\sum_{r=1}^{n} r\right)^2$ for *all n*; but it is true and this will be proved in Chapter 6]

e The statement is not true.

$$\sum_{r=1}^{n}\left(3r^2+4\right) = \left\{3\times 1^2+4\right\}+\left\{3\times 2^2+4\right\}+\left\{3\times 3^2+4\right\}+\ldots+\left\{3n^2+4\right\}$$

$$= 3\{1^2+2^2+3^2+\ldots+n^2\}+4n$$

$$3\sum_{r=1}^{n} r^2+4 = 3\{1^2+2^2+3^2+\ldots+n^2\}+4$$

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Series
Exercise A, Question 4

Question:

Express these series using Σ notation.

a $3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$

b $1 + 8 + 27 + 64 + 125 + 216 + 243 + 512$

c $11 + 21 + 35 + \ldots + (2n^2 + 3)$

d $11 + 21 + 35 + \ldots + (2n^2 - 4n + 5)$

e $3 \times 5 + 5 \times 7 + 7 \times 9 + \ldots + (2r - 1)(2r + 1) + \ldots$ to k terms.

Solution:

Answers are not unique (two examples are given, and any letter may be used for r)

a $\sum_{r=3}^{10} r,\quad \sum_{r=1}^{8}(r+2)$

b $\sum_{r=1}^{8} r^3,\quad \sum_{r=2}^{9}(r-1)^3$

c $\sum_{r=2}^{n}(2r^2+3),\quad \sum_{r=3}^{n+1}(2r^2-4r+5)$

d $\sum_{r=3}^{n}(2r^2-4r+5),\quad \sum_{r=2}^{n-1}(2r^2+3).$

e $\sum_{r=2}^{k+1}(2r-1)(2r+1),\quad \sum_{r=1}^{k}(2r+1)(2r+3)$

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Series
Exercise B, Question 1

Question:

Use the result for $\sum_{r=1}^{n} r$ to calculate

a $\sum_{r=1}^{36} r$

b $\sum_{r=1}^{99} r$

c $\sum_{p=10}^{55} p$

d $\sum_{r=100}^{200} r$

e $\sum_{r=1}^{k} r + \sum_{r=k+1}^{80} r$, where $k < 80$.

Solution:

a $\frac{36 \times 37}{2} = 666$

b $\frac{99 \times 100}{2} = 4950$

c $\sum_{p=1}^{55} p - \sum_{p=1}^{9} p = \frac{55 \times 56}{2} - \frac{9 \times 10}{2} = 1540 - 45 = 1495$

d $\sum_{r=1}^{200} r - \sum_{r=1}^{99} r = \frac{200 \times 201}{2} - \frac{99 \times 100}{2} = 20100 - 4950 = 15150$

e $\sum_{r=1}^{k} r + \sum_{r=k+1}^{80} r = \sum_{r=1}^{80} r = \frac{80 \times 81}{2} = 3240$

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Series
Exercise B, Question 2

Question:

Given that $\sum_{r=1}^{n} r = 528$,

a show that $n^2 + n - 1056 = 0$

b find the value of n.

Solution:

a $\frac{n}{2}(n+1) = 528 \Rightarrow n(n+1) = 1056 \Rightarrow n^2 + n - 1056 = 0$

b Factorising: $(n-32)(n+33) = 0$ (or use "the formula") $\Rightarrow n = 32$, as n cannot be negative.

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Series
Exercise B, Question 3

Question:

a Find $\sum_{k=1}^{2n-1} k$.

b Hence show that $\sum_{k=n+1}^{2n-1} k = \frac{3n}{2}(n-1), n \geq 2$.

Solution:

a $\frac{(2n-1)\{(2n-1)+1\}}{2} = \frac{(2n-1)(2n)}{2} = n(2n-1)$

b

$$\sum_{k=1}^{2n-1} k - \sum_{k=1}^{n} k = n(2n-1) - \frac{n}{2}(n+1) = \frac{n}{2}\{2(2n-1)-(n+1)\} = \frac{n}{2}(3n-3)$$

$$= \frac{3n}{2}(n-1)$$

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Series
Exercise B, Question 4

Question:

Show that $\sum_{r=k-1}^{2k} r = \frac{(k+2)(3k-1)}{2}, k \geq 1$

Solution:

$$\sum_{r=1}^{2k} r - \sum_{r=1}^{k-2} r = \frac{2k}{2}(2k+1) - \frac{(k-2)}{2}(k-1) = \frac{(4k^2+2k)-(k^2-3k+2)}{2}$$

$$= \frac{3k^2+5k-2}{2} = \frac{(3k-1)(k+2)}{2}$$

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Series
Exercise B, Question 5

Question:

a Show that $\sum_{r=1}^{n^2} r - \sum_{r=1}^{n} r = \frac{n(n^3-1)}{2}$.

b Hence evaluate $\sum_{r=10}^{81} r$.

Solution:

a $\frac{n^2(n^2+1)}{2} - \frac{n(n+1)}{2} = \frac{n}{2}\left\{n(n^2+1)-(n+1)\right\} = \frac{n}{2}(n^3-1)$

b $\sum_{r=10}^{81} r = \sum_{r=1}^{9^2} r - \sum_{r=1}^{9} r = \frac{9}{2}(9^3-1)$ [using part (a)] $= 3276$.

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Series
Exercise C, Question 1

Question:

(In this exercise use the results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} 1$.)

Calculate the sum of the series:

a $\sum_{r=1}^{55}(3r-1)$

b $\sum_{r=1}^{90}(2-7r)$

c $\sum_{r=1}^{46}(9+2r)$

Solution:

a $3\sum_{r=1}^{55} r-\sum_{r=1}^{55} 1=3\times\frac{55\times 56}{2}-55=4565$

b $2\sum_{r=1}^{90} 1-7\sum_{r=1}^{90} r=2\times 90-7\times\frac{90\times 91}{2}=-28485$

c $9\sum_{r=1}^{46} 1+2\sum_{r=1}^{46} r=9\times 46+2\times\frac{46\times 47}{2}=2576$

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Series
Exercise C, Question 2

Question:

Show that

a $\sum_{r=1}^{n}(3r+2)=\frac{n}{2}(3n+7)$

b $\sum_{i=1}^{2n}(5i-4)=n(10n-3)$

c $\sum_{r=1}^{n+2}(2r+3)=(n+2)(n+6)$

d

$\sum_{p=3}^{n}(4p+5)=(2n+11)(n-2)$

Solution:

a $3\sum_{r=1}^{n}r+2\sum_{r=1}^{n}1=3\times\frac{n}{2}(n+1)+2n=\frac{n}{2}(3n+3+4)=\frac{n}{2}(3n+7)$

b $5\sum_{i=1}^{2n}i-4\sum_{i=1}^{2n}1=5\times\frac{2n}{2}(2n+1)-4(2n)=n(10n+5-8)=n(10n-3)$

c $2\sum_{r=1}^{n+2}r+3\sum_{r=1}^{n+2}1=2\times\frac{(n+2)}{2}(n+3)+3(n+2)=(n+2)(n+3+3)=(n+2)(n+6)$

d

$$\left\{4\sum_{p=1}^{n}p+5\sum_{p=1}^{n}1\right\}-\sum_{p=1}^{2}(4p+5)=\left\{4\times\frac{n}{2}(n+1)+5n\right\}-(9+13)$$

$$=2n^2+7n-22=(2n+11)(n-2)$$

Solutionbank FP1
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Series
Exercise C, Question 3

Question:

a Show that $\sum_{r=1}^{k}(4r-5)=2k^2-3k$.

b Find the smallest value of k for which $\sum_{r=1}^{k}(4r-5)>4850$.

Solution:

a $4\sum_{r=1}^{k} r-5\sum_{r=1}^{k}1=4\times\frac{k}{2}(k+1)-5k=2k^2-3k$

b $2k^2-3k>4850 \Rightarrow 2k^2-3k-4850>0 \Rightarrow (2k+97)(k-50)>0$,

so $k>50$ [k is positive] $\Rightarrow k=51$

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Series
Exercise C, Question 4

Question:

Given that $u_r = ar + b$ and $\sum_{r=1}^{n} u_r = \frac{n}{2}(7n + 1)$, find the constants a and b.

Solution:

$$\sum_{r=1}^{n}(ar + b) = \frac{an}{2}(n + 1) + bn = \frac{an^2 + (a + 2b)n}{2}$$

Comparing with $\frac{7n^2 + n}{2} \Rightarrow a = 7$ and $a + 2b = 1$

So $a = 7$, $b = -3$

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Series
Exercise C, Question 5

Question:

a Show that $\sum_{r=1}^{4n-1}(1+3r)=24n^2-2n-1\ n\geq 1$.

b Hence calculate $\sum_{r=1}^{99}(1+3r)$.

Solution:

a $\sum_{r=1}^{4n-1}1+3\sum_{r=1}^{4n-1}r=(4n-1)+3\times\frac{(4n-1)(4n)}{2}=(4n-1)(1+6n)=24n^2-2n-1$

b Substituting $n=25$ into above result gives 14949

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Series
Exercise C, Question 6

Question:

Show that $\sum_{r=1}^{2k+1}(4-5r) = -(2k+1)(5k+1), k \geq 0$

Solution:

$$4\sum_{r=1}^{2k+1}1 - 5\sum_{r=1}^{2k+1}r = 4(2k+1) - 5\frac{(2k+1)}{2}(2k+2) \quad = (2k+1)\{4-5(k+1)\}$$

$$= (2k+1)(-1-5k) = -(2k+1)(5k+1)$$

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Series
Exercise D, Question 1

Question:

Verify that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$ is true for $n = 1, 2$ and 3.

Solution:

For $n = 1$, $\sum_{r=1}^{n} r^2 = 1^2 = 1$, $\frac{n}{6}(n+1)(2n+1) = \frac{1}{6}(1+1)(2+1) = 1$

For $n = 2$, $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 = 5$, $\frac{n}{6}(n+1)(2n+1) = \frac{2}{6}(2+1)(4+1) = 5$

For $n = 3$, $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 = 14$, $\frac{n}{6}(n+1)(2n+1) = \frac{3}{6}(3+1)(6+1) = 14$

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Series
Exercise D, Question 2

Question:

a By writing out each series, evaluate $\sum_{r=1}^{n} r$ for $n = 1, 2, 3$ and 4.

b By writing out each series, evaluate $\sum_{r=1}^{n} r^3$ for $n = 1, 2, 3$ and 4.

c What do you notice about the corresponding results for each value of n ?

Solution:

a $\sum_{r=1}^{1} r = 1$; $\sum_{r=1}^{2} r = 1 + 2 = 3$; $\sum_{r=1}^{3} r = 1 + 2 + 3 = 6$; $\sum_{r=1}^{4} r = 1 + 2 + 3 + 4 = 10$

b $\sum_{r=1}^{1} r^3 = 1$; $\sum_{r=1}^{2} r^3 = 1^3 + 2^3 = 9$; $\sum_{r=1}^{3} r^3 = 1^3 + 2^3 + 3^3 = 36$; $\sum_{r=1}^{4} r^3 = 1^3 + 2^3 + 3^3 + 4^3 = 100$

c The results for (b) are the square of the results for (a)

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Series
Exercise D, Question 3

Question:

Using the appropriate formula, evaluate

a $\sum_{r=1}^{100} r^2$

b $\sum_{r=20}^{40} r^2$

c $\sum_{r=1}^{30} r^3$

d $\sum_{r=25}^{45} r^3$

Solution:

a $\frac{100}{6} \times 101 \times 201 = 338350$

b $\sum_{r=1}^{40} r^2 - \sum_{r=1}^{19} r^2 = \frac{40}{6} \times 41 \times 81 - \frac{19}{6} \times 20 \times 39 = 22140 - 2470 = 19670$

c $\frac{30^2 \times 31^2}{4} = 216225$

d $\sum_{r=1}^{45} r^3 - \sum_{r=1}^{24} r^3 = \frac{45^2 \times 46^2}{4} - \frac{24^2 \times 25^2}{4} = 1071225 - 90000 = 981225$

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Series
Exercise D, Question 4

Question:

Use the formula for $\sum_{r=1}^{n} r^2$ or $\sum_{r=1}^{n} r^3$ to find the sum of

a $1^2 + 2^2 + 3^2 + 4^2 + \ldots + 52^2$

b $2^3 + 3^3 + 4^3 + \ldots + 40^3$

c $26^2 + 27^2 + 28^2 + 29^2 + \ldots + 100^2$

d $1^2 + 2^2 + 3^2 + \ldots + (k+1)^2$

e $1^3 + 2^3 + 3^3 + \ldots + (2n-1)^3$

Solution:

a $\sum_{r=1}^{52} r^2 = \frac{52}{6} \times 53 \times 105 = 48230$

b $\sum_{r=1}^{40} r^3 - 1 = \frac{40^2 \times 41^2}{4} - 1 = 672399$

c $\sum_{r=1}^{100} r^2 - \sum_{r=1}^{25} r^2 = \frac{100}{6} \times 101 \times 201 - \frac{25}{6} \times 26 \times 51 = 338350 - 5525 = 332825$

d $\sum_{r=1}^{k+1} r^2 = \frac{(k+1)}{6}(k+2)(2k+3)$

e $\sum_{r=1}^{2n-1} r^3 = \frac{(2n-1)^2(2n)^2}{4} = n^2(2n-1)^2$

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Series
Exercise D, Question 5

Question:

For each of the following series write down, in terms of n, the sum, giving the result in its simplest form

a $\sum_{r=1}^{2n} r^2$

b $\sum_{r=1}^{n^2-1} r^2$

c $\sum_{i=1}^{2n-1} i^2$

d $\sum_{r=1}^{n+1} r^3$

e $\sum_{k=n+1}^{3n} k^3, n > 0.$

Solution:

a $\frac{(2n)}{6}(2n+1)(4n+1) = \frac{n}{3}(2n+1)(4n+1)$

b $\frac{(n^2-1)n^2(2n^2-1)}{6}$

c $\frac{(2n-1)}{6}(2n)[2(2n-1)+1] = \frac{(2n-1)}{6}(2n)(4n-1) = \frac{n}{3}(2n-1)(4n-1)$

d $\frac{(n+1)^2(n+2)^2}{4}$

e

$$\begin{aligned}\sum_{r=1}^{3n} k^3 - \sum_{r=1}^{n} k^3 &= \frac{(3n)^2(3n+1)^2}{4} - \frac{n^2(n+1)^2}{4} = \frac{n^2}{4}\{9(3n+1)^2-(n+1)^2\} \\ &= \frac{n^2}{4}\{3(3n+1)-(n+1)\}\{3(3n+1)+(n+1)\}[\text{using } a^2-b^2=(a-b)(a+b)] \\ &= \frac{n^2}{4}\{(8n+2)(10n+4)\} \\ &= n^2(4n+1)(5n+2)\end{aligned}$$

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Series
Exercise D, Question 6

Question:

Show that

a $\sum_{r=2}^{n} r^2 = \frac{1}{6}(n-1)(2n^2+5n+6)$

b $\sum_{r=n}^{2n} r^2 = \frac{n}{6}(n+1)(14n+1)$

Solution:

a $\frac{n}{6}(n+1)(2n+1) - 1 = \frac{2n^3+3n^2+n-6}{6} = \frac{(n-1)(2n^2+5n+6)}{6}$ [use factor theorem]

b

$$\sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2 = \frac{2n}{6}(2n+1)(4n+1) - \frac{(n-1)}{6}n(2n-1)$$

$$= \frac{n}{6}\{2(2n+1)(4n+1) - (n-1)(2n-1)\}$$

$$\frac{n}{6}\{(16n^2+12n+2) - (2n^2-3n+1)\} = \frac{n}{6}(14n^2+15n+1)$$

$$= \frac{n}{6}(14n+1)(n+1)$$

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Series
Exercise D, Question 7

Question:

a Show that $\sum_{k=n}^{2n} k^3 = \frac{3n^2(n+1)(5n+1)}{4}$

b Find $\sum_{k=30}^{60} k^3$.

Solution:

a

$$\sum_{k=1}^{2n} k^3 - \sum_{k=1}^{n-1} k^3 = \frac{(2n)^2(2n+1)^2}{4} - \frac{(n-1)^2 n^2}{4}$$

$$= \frac{n^2}{4}\{4(2n+1)^2 - (n-1)^2\}$$

$$= \frac{n^2}{4}[\{2(2n+1)+(n-1)\}\{2(2n+1)-(n-1)\} \quad \text{“Difference of two squares”}$$

$$= \frac{n^2}{4}(5n+1)(3n+3) = \frac{3n^2}{4}(5n+1)(n+1)$$

b Substituting $n = 30$ into (a) gives 3 159 675

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Series
Exercise D, Question 8

Question:

a Show that $\sum_{r=1}^{2n} r^3 = n^2(2n+1)^2$.

b By writing out the series for $\sum_{r=1}^{n}(2r)^3$, show that $\sum_{r=1}^{n}(2r)^3 = 8\sum_{r=1}^{n} r^3$.

c Show that $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$ can be written as $\sum_{r=1}^{2n} r^3 - \sum_{r=1}^{n}(2r)^3$.

d Hence show that the sum of the cubes of the first n odd natural numbers, $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$, is $n^2(2n^2-1)$.

Solution:

a $\sum_{r=1}^{2n} r^3 = \frac{(2n)^2(2n+1)^2}{4} = n^2(2n+1)^2$.

b $\sum_{r=1}^{n}(2r)^3 = 2^3 + 4^3 + 6^3 + \ldots + (2n)^3 = 2^3\{1^3 + 2^3 + 3^3 + \ldots + n^3\} = 8\sum_{r=1}^{n} r^3$.

c

$$\begin{aligned}1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 &= \{1^3 + 2^3 + 3^3 + \ldots + (2n-1)^3 + (2n)^3\} - \{2^3 + 4^3 + 6^3 + \ldots + (2n)^3\}\\ &= \sum_{r=1}^{2n} r^3 - \sum_{r=1}^{n}(2r)^3.\end{aligned}$$

d Using the results in parts (b) and (c), $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = \sum_{r=1}^{2n} r^3 - 8\sum_{r=1}^{n} r^3$

$$\begin{aligned}&= n^2(2n+1)^2 - 8\sum_{r=1}^{n} r^3 \text{ (using(a))}\\ &= n^2(2n+1)^2 - \frac{8n^2(n+1)^2}{4}\\ &= n^2[(2n+1)^2 - 2(n+1)^2]\\ &= n^2[(4n^2+4n+1) - 2(n^2+2n+1)]\\ &= n^2(2n^2-1)\end{aligned}$$

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Series
Exercise E, Question 1

Question:

Use the formulae for $\sum_{r=1}^{n} r^3$, $\sum_{r=1}^{n} r^2$, $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} 1$, where appropriate, to find

a $\sum_{m=1}^{30} (m^2 - 1)$

b $\sum_{r=1}^{40} r(r+4)$

c $\sum_{r=1}^{80} r(r^2+3)$

d $\sum_{r=11}^{35} (r^3-2)$.

Solution:

a $\sum_{m=1}^{30} m^2 - 30 = \frac{30 \times 31 \times 61}{6} - 30 = 9425$

b $\sum_{r=1}^{40} r^2 + 4\sum_{r=1}^{40} r = \frac{40 \times 41 \times 81}{6} + 4 \times \frac{40 \times 41}{2} = 22140 + 3280 = 25420$

c $\sum_{r=1}^{80} r^3 + 3\sum_{r=1}^{80} r = \frac{80^2 \times 81^2}{4} + 3 \times \frac{80 \times 81}{2} = 10497600 + 9720 = 10507\,320$

d $\sum_{r=1}^{35} (r^3-2) - \sum_{r=1}^{10} (r^3-2) = \sum_{r=1}^{35} r^3 - 2(35) - \left[\sum_{r=1}^{10} r^3 - 2(10)\right]$

$\sum_{r=1}^{35} r^3 - \sum_{r=1}^{10} r^3 - 2(35-10) = \frac{35^2 \times 36^2}{4} - \frac{10^2 \times 11^2}{4} - 50 = 396900 - 3025 - 50 = 393825.$

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Series
Exercise E, Question 2

Question:

Use the formulae for $\sum_{r=1}^{n} r^3$, $\sum_{r=1}^{n} r^2$, and $\sum_{r=1}^{n} r$, where appropriate, to find

a $\sum_{r=1}^{n} (r^2 + 4r)$

b $\sum_{r=1}^{n} r(2r^2 - 1)$

c $\sum_{r=1}^{2n} r^2(1 + r)$, giving your answer in its simplest form.

Solution:

a $\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r = \frac{n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} = \frac{n(n+1)\{(2n+1)+12\}}{6} = \frac{n}{6}(n+1)(2n+13)$

b $2\sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r = \frac{2n^2(n+1)^2}{4} - \frac{n(n+1)}{2} = \frac{n(n+1)\{n(n+1)-1\}}{2} = \frac{n}{2}(n+1)(n^2+n-1)$

c

$$\sum_{r=1}^{2n} r^2 + \sum_{r=1}^{2n} r^3 = \frac{2n(2n+1)(4n+1)}{6} + \frac{(2n)^2(2n+1)^2}{4} = \frac{n(2n+1)\{(4n+1)+3n(2n+1)\}}{3}$$

$$= \frac{n}{3}(2n+1)(6n^2+7n+1) = \frac{n}{3}(n+1)(2n+1)(6n+1)$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Series
Exercise E, Question 3

Question:

a Write out $\sum_{r=1}^{n} r(r+1)$ as a sum, showing at least the first three terms and the final term.

b Use the results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to calculate

$1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + 5 \times 6 + \ldots + 60 \times 61.$

Solution:

a $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n(n+1)$

b Putting $n = 60$: $\sum_{r=1}^{60} r^2 + \sum_{r=1}^{60} r = \frac{60 \times 61 \times 121}{6} + \frac{60 \times 61}{2} = 73810 + 1830 = 75640$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Series
Exercise E, Question 4

Question:

a Show that $\sum_{r=1}^{n}(r+2)(r+5)=\frac{n}{3}(n^2+12n+41)$.

b Hence calculate $\sum_{r=10}^{50}(r+2)(r+5)$.

Solution:

a

$$\begin{aligned}\sum_{r=1}^{n}(r^2+7r+10) &= \sum_{r=1}^{n}r^2+7\sum_{r=1}^{n}r+10\sum_{r=1}^{n}1\\ &= \frac{n}{6}(n+1)(2n+1)+7\frac{n}{2}(n+1)+10n\\ &= \frac{n}{6}\{(2n^2+3n+1)+21(n+1)+60\}\\ &= \frac{n}{6}(2n^2+24n+82)=\frac{n}{3}(n^2+12n+41)\end{aligned}$$

b Substituting $n = 50$ and $n = 9$ in the formula in (a), and subtracting, gives 51 660.

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Series
Exercise E, Question 5

Question:

a Show that $\sum_{r=2}^{n}(r-1)r(r+1)=\frac{(n-1)n(n+1)(n+2)}{4}$.

b Hence find the sum of the series $13\times14\times15+14\times15\times16+15\times16\times17+\ldots+44\times45\times46$.

Solution:

a

$$\sum_{r=2}^{n}(r^3-r)=\sum_{r=1}^{n}(r^3-r)=\sum_{r=1}^{n}r^3-\sum_{r=1}^{n}r=\frac{n^2(n+1)^2}{4}-\frac{n}{2}(n+1)$$
$$=\frac{n(n+1)}{4}(n^2+n-2)$$
$$=\frac{n}{4}(n+1)\{n^2+n-2\}$$
$$=\frac{n}{4}(n+1)(n+2)(n-1)=\frac{(n-1)n(n+1)(n+2)}{4}$$

b $\sum_{r=14}^{45}(r-1)r(r+1)=\sum_{r=2}^{45}(r-1)r(r+1)-\sum_{r=2}^{13}(r-1)r(r+1)=\frac{44\times45\times46\times47}{4}-\frac{12\times13\times14\times15}{4}$

= 1 062 000

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Series
Exercise E, Question 6

Question:

Find the following sums, and check your results for the cases $n = 1, 2$ and 3.

a $\sum_{r=1}^{n} (r^3 - 1)$

b $\sum_{r=1}^{n} (2r - 1)^2$

c $\sum_{r=1}^{n} r(r + 1)^2$

Solution:

a $\sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} 1 = \frac{n^2(n+1)^2}{4} - n = \frac{n}{4}\{n(n+1)^2 - 4\} = \frac{n}{4}(n^3 + 2n^2 + n - 4)$

When $n = 1$: $\sum_{r=1}^{1} (r^3 - 1) = 0;$ $\frac{n}{4}(n^3 + 2n^2 + n - 4) = \frac{1 \times 0}{4} = 0$

When $n = 2$: $\sum_{r=1}^{2} (r^3 - 1) = 0 + 7 = 7;$ $\frac{n}{4}(n^3 + 2n^2 + n - 4) = \frac{2 \times 14}{4} = 7$

When $n = 3$: $\sum_{r=1}^{3} (r^3 - 1) = 0 + 7 + 26 = 33;$ $\frac{n}{4}(n^3 + 2n^2 + n - 4) = \frac{3 \times 44}{4} = 33$

b

$$\begin{aligned}\sum_{r=1}^{n} (4r^2 - 4r + 1) &= 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 = \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n \\ &= \frac{n}{3}\left\{2(2n^2 + 3n + 1) - 6(n+1) + 3\right\} = \frac{n}{3}(4n^2 - 1)\end{aligned}$$

When $n = 1$: $\sum_{r=1}^{1} (4r^2 - 4r + 1) = 1;$ $\frac{n}{3}(4n^2 - 1) = \frac{1 \times 3}{3} = 1$

When $n = 2$: $\sum_{r=1}^{2} (4r^2 - 4r + 1) = 1 + 9 = 10;$ $\frac{n}{3}(4n^2 - 1) = \frac{2 \times 15}{3} = 10$

When $n = 3$: $\sum_{r=1}^{3} (4r^2 - 4r + 1) = 1 + 9 + 25 = 35;$ $\frac{n}{3}(4n^2 - 1) = \frac{3 \times 35}{3} = 35$

c

$$\sum_{r=1}^{n}(r^3+2r^2+r) = \sum_{r=1}^{n} r^3 + 2\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r = \frac{n^2(n+1)^2}{4} + \frac{2n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{12}\{3n(n+1)+4(2n+1)+6\} = \frac{n(n+1)}{12}\{3n^2+11n+10\}$$

$$= \frac{n}{12}(n+1)(n+2)(3n+5)$$

When $n = 1$: $\sum_{r=1}^{1} r(r+1)^2 = 1\times 4 = 4;\quad \frac{n}{12}(n+1)(n+2)(3n+5) = \frac{1\times 2\times 3\times 8}{12} = 4$

When $n = 2$: $\sum_{r=1}^{2} r(r+1)^2 = 4+2\times 9 = 22;\quad \frac{n}{12}(n+1)(n+2)(3n+5) = \frac{2\times 3\times 4\times 11}{12} = 22$

When $n = 3$: $\sum_{r=1}^{3} r(r+1)^2 = 22+3\times 16 = 70;\quad \frac{n}{12}(n+1)(n+2)(3n+5) = \frac{3\times 4\times 5\times 14}{12} = 70$

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Series
Exercise E, Question 7

Question:

a Show that $\sum_{r=1}^{n} r^2(r-1) = \frac{n}{12}(n^2-1)(3n+2)$.

b Deduce the sum of $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \ldots + 30 \times 31^2$.

Solution:

a

$$\begin{aligned}\sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^2 &= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} \\ &= \frac{n(n+1)}{12}\{3n(n+1) - 2(2n+1)\} \\ &= \frac{n(n+1)}{12}(3n^2 - n - 2) \\ &= \frac{n(n+1)(n-1)(3n+2)}{12} = \frac{n(n^2-1)(3n+2)}{12}\end{aligned}$$

b As $\sum_{r=2}^{31} r^2(r-1) = \sum_{r=1}^{31} r^2(r-1)$, substitute $n = 31$ in (a); sum = 235 600

Solutionbank FP1
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Series
Exercise E, Question 8

Question:

a Show that $\sum_{r=2}^{n}(r-1)(r+1)=\frac{n}{6}(2n+5)(n-1)$.

b Hence sum the series $1\times3+2\times4+3\times5+\ldots+35\times37$.

Solution:

a [$\sum_{r=2}^{n}(r^2-1)=\sum_{r=1}^{n}(r^2-1)$ as when $r=1$ the term is zero]

$$\sum_{r=1}^{n}(r^2-1)=\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}1 = \frac{n}{6}(n+1)(2n+1)-n$$
$$=\frac{n}{6}\{(2n^2+3n+1)-6\}$$
$$=\frac{n}{6}(2n^2+3n-5)$$
$$=\frac{n}{6}(2n+5)(n-1)$$

b $1\times3+2\times4+3\times5+\ldots+35\times37=\sum_{r=1}^{36}(r-1)(r+1)$

Substituting $n=36$ into result in (a) gives 16 170

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Series
Exercise E, Question 9

Question:

a Write out the series defined by $\sum_{r=7}^{12} r(2+3r)$, and hence find its sum.

b Show that $\sum_{r=n+1}^{2n} r(2+3r) = \frac{n}{2}(14n^2+15n+3)$.

c By substituting the appropriate value of n into the formula in **b**, check that your answer for **a** is correct.

Solution:

a $7\times 23+8\times 26+9\times 29+10\times 32+11\times 35+12\times 38 = 1791$.

b $\sum_{r=n+1}^{2n}(2r+3r^2) = \sum_{r=1}^{2n}(2r+3r^2) - \sum_{r=1}^{n}(2r+3r^2)$

$$\begin{aligned}\sum_{r=1}^{n}(2r+3r^2) = 2\sum_{r=1}^{n} r + 3\sum_{r=1}^{n} r^2 &= n(n+1)+\frac{n}{2}(n+1)(2n+1)\\ &= \frac{n}{2}(n+1)\{2+(2n+1)\}\\ &= \frac{n}{2}(n+1)(2n+3)\end{aligned}$$

$$\Rightarrow \qquad \sum_{r=1}^{2n}(2r+3r^2) = n(2n+1)(4n+3)$$

$$\begin{aligned}\sum_{r=n+1}^{2n}(2r+3r^2) &= n(2n+1)(4n+3)-\frac{n}{2}(n+1)(2n+3)\\ &= \frac{n}{2}\{2(2n+1)(4n+3)-(n+1)(2n+3)\}\\ &= \frac{n}{2}\{(16n^2+20n+6)-(2n^2+5n+3)\}\\ &= \frac{n}{2}(14n^2+15n+3)\end{aligned}$$

c Substituting $n = 6$ gives 1791

Solutionbank FP1
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Series
Exercise E, Question 10

Question:

Find the sum of the series $1 \times 1 + 2 \times 3 + 3 \times 5 + \ldots$ to n terms.

Solution:

Series can be written as $\sum_{r=1}^{n} r(2r-1)$

$$\sum_{r=1}^{n} r(2r-1) = 2\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r \quad = 2 \times \frac{n}{6}(n+1)(2n+1) - \frac{n}{2}(n+1)$$

$$= \frac{n(n+1)\{2(2n+1)-3\}}{6}$$

$$= \frac{n(n+1)(4n-1)}{6}$$

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Series
Exercise F, Question 1

Question:

a Write down the first three terms and the last term of the series given by $\sum_{r=1}^{n}(2r+3^r)$.

b Find the sum of this series.

c Verify that your result in **b** is correct for the cases $n = 1, 2$ and 3.

Solution:

a $(2+3)+(4+3^2)+(6+3^3)+\ldots+(2n+3^n)\quad[=5+13+33+\ldots+(2n+3^n)]$

b $\sum_{r=1}^{n}(2r+3^r)=2\sum_{r=1}^{n}r+\sum_{r=1}^{n}3^r=n(n+1)+\frac{3}{2}(3^n-1)\quad[\text{AP}+\text{GP}]$

c

$n=1$: (b) gives $2+3=5$, agrees with (a)

$n=2$: (b) gives $6+12=18$, agrees with (a)

$n=3$: (b) gives $12+39=51$, agrees with (a)

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Series
Exercise F, Question 2

Question:

Find

a $\sum_{r=1}^{50}(7r+5)$

b $\sum_{r=1}^{40}(2r^2-1)$

c $\sum_{r=33}^{75} r^3$.

Solution:

a $7\sum_{r=1}^{50} r+5\sum_{r=1}^{50} 1=\frac{7\times 50\times 51}{2}+5(50)=9175$

b $2\sum_{r=1}^{40} r^2-\sum_{r=1}^{40} 1=\frac{40(41)(81)}{3}-40=44\ 240$

c $\sum_{r=1}^{75} r^3-\sum_{r=1}^{32} r^3=\frac{75^2\times 76^2}{4}-\frac{32^2\times 33^2}{4}=7\ 843\ 716$

Solutionbank FP1
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Series
Exercise F, Question 3

Question:

Given that $\sum_{r=1}^{n} U_r = n^2 + 4n$,

a find $\sum_{r=1}^{n-1} U_r$.

b Deduce an expression for U_n.

c Find $\sum_{r=n}^{2n} U_r$.

Solution:

a Replacing n with $(n-1)$ gives $(n-1)^2 + 4(n-1) = n^2 + 2n - 3$

b $U_n = \sum_{r=1}^{n} U_r - \sum_{r=1}^{n-1} U_r = n^2 + 4n - (n^2 + 2n - 3) = 2n + 3$

c $\sum_{r=1}^{2n} U_r - \sum_{r=1}^{n-1} U_r = (4n^2 + 8n) - (n^2 + 2n - 3) = 3n^2 + 6n + 3 = 3(n+1)^2$

Solutionbank FP1
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Series
Exercise F, Question 4

Question:

Evaluate $\sum_{r=1}^{30} r(3r-1)$

Solution:

$$3\sum_{r=1}^{30} r^2 - \sum_{r=1}^{30} r = \frac{3\times 30\times 31\times 61}{6} - \frac{30\times 31}{2} = 28365 - 465 = 27900$$

Solutionbank FP1
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Series
Exercise F, Question 5

Question:

Find $\sum_{r=1}^{n} r^2(r-3)$.

Solution:

$$\begin{aligned}\sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r^2 &= \frac{n^2}{4}(n+1)^2 - \frac{n}{2}(n+1)(2n+1)\\ &= \frac{n}{4}(n+1)\{n(n+1)-2(2n+1)\}\\ &= \frac{n}{4}(n+1)(n^2-3n-2)\end{aligned}$$

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Series
Exercise F, Question 6

Question:

Show that $\sum_{r=1}^{2n}(2r-1)^2 = \frac{2n}{3}(16n^2-1)$.

Solution:

$$\begin{aligned}4\sum_{r=1}^{2n} r^2 - 4\sum_{r=1}^{2n} r + \sum_{r=1}^{2n} 1 &= \frac{4}{3}n(2n+1)(4n+1) - 4n(2n+1) + 2n \\ &= \frac{n}{3}\{4(2n+1)(4n+1) - 12(2n+1) + 6\} \\ &= \frac{n}{3}\{32n^2 + 24n + 4 - 12(2n+1) + 6\} \\ &= \frac{n}{3}(32n^2 - 2) \\ &= \frac{2n}{3}(16n^2 - 1)\end{aligned}$$

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Series
Exercise F, Question 7

Question:

a Show that $\sum_{r=1}^{n} r(r+2) = \frac{n}{6}(n+1)(2n+7)$.

b Using this result, or otherwise, find in terms of n, the sum of $3\log 2 + 4\log 2^2 + 5\log 2^3 + \ldots + (n+2)\log 2^n$.

Solution:

a

$$\begin{aligned}\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r &= \frac{n}{6}(n+1)(2n+1) + 2\frac{n}{2}(n+1)\\ &= \frac{n}{6}(n+1)\{(2n+1)+6\}\\ &= \frac{n}{6}(n+1)(2n+7)\end{aligned}$$

b

$$\begin{aligned}\text{The series is : } \sum_{r=1}^{n}(r+2)\log 2^r &= \sum_{r=1}^{n} r(r+2)\log 2 \quad \text{as } \log 2^r = r\ \log 2\\ &= \log 2\sum_{r=1}^{n} r(r+2) \quad \text{as } \log 2 \text{ is a constant}\\ &= \frac{n}{6}(n+1)(2n+7)\ \log 2\end{aligned}$$

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Series
Exercise F, Question 8

Question:

Show that $\sum_{r=n}^{2n} r^2 = \frac{n}{6}(n+1)(an+b)$, where a and b are constants to be found.

Solution:

$$\sum_{r=n}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2 = \frac{(2n)(2n+1)(4n+1)}{6} - \frac{(n-1)n(2n-1)}{6}$$
$$= \frac{n}{6}\{2(8n^2+6n+1)-(2n^2-3n+1)\}$$
$$= \frac{n}{6}(14n^2+15n+1)$$
$$= \frac{n}{6}(n+1)(14n+1) \qquad a = 14, \quad b = 1$$

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Series
Exercise F, Question 9

Question:

a Show that $\sum_{r=1}^{n}(r^2 - r - 1) = \frac{n}{3}(n-2)(n+2)$.

b Hence calculate $\sum_{r=10}^{40}(r^2 - r - 1)$.

Solution:

a

$$\begin{aligned}\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 &= \frac{n}{6}(n+1)(2n+1) - \frac{n}{2}(n+1) - n \\ &= \frac{n}{6}\{(n+1)(2n+1) - 3(n+1) - 6\} \\ &= \frac{n}{6}(2n^2 - 8) \\ &= \frac{n}{3}(n^2 - 4) \\ &= \frac{n}{3}(n-2)(n+2)\end{aligned}$$

b $\sum_{r=10}^{40}(r^2 - r - 1) = \sum_{r=1}^{40}(r^2 - r - 1) - \sum_{r=1}^{9}(r^2 - r - 1)$

Substitute $n = 40$ and $n = 9$ into the result for part (a), and subtract.

The result is 21280 – 230 = 21049

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Series
Exercise F, Question 10

Question:

a Show that $\sum_{r=1}^{n} r(2r^2+1) = \frac{n}{2}(n+1)(n^2+n+1)$.

b Hence calculate $\sum_{r=26}^{58} r(2r^2+1)$.

Solution:

a

$$\begin{aligned} 2\sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r &= \frac{n^2(n+1)^2}{2} + \frac{n}{2}(n+1) \\ &= \frac{n}{2}(n+1)\{n(n+1)+1\} \\ &= \frac{n}{2}(n+1)(n^2+n+1) \end{aligned}$$

b Substitute $n = 58$ and $n = 25$ into the result for (a), and subtract. The result = 5654178.

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Series
Exercise F, Question 11

Question:

Find

a $\sum_{r=1}^{n} r(3r-1)$

b $\sum_{r=1}^{n} (r+2)(3r+5)$

c $\sum_{r=1}^{n} (2r^3 - 2r + 1)$.

Solution:

a $3\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r = \frac{n(n+1)(2n+1)}{2} - \frac{n(n+1)}{2} = \frac{n(n+1)}{2}(2n+1-1) = n^2(n+1)$

b

$$\begin{aligned}3\sum_{r=1}^{n} r^2 + 11\sum_{r=1}^{n} r + 10\sum_{r=1}^{n} 1 &= \frac{n(n+1)(2n+1)}{2} + \frac{11n(n+1)}{2} + 10n \\ &= \frac{n}{2}\{(2n^2+3n+1) + 11(n+1) + 20\} \\ &= \frac{n}{2}(2n^2+14n+32) = n(n^2+7n+16)\end{aligned}$$

c

$$\begin{aligned}2\sum_{r=1}^{n} r^3 - 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 &= \frac{n^2(n+1)^2}{2} - n(n+1) + n \\ &= \frac{n}{2}\{n(n+1)^2 - 2(n+1) + 2] \\ &= \frac{n}{2}\{n(n+1)^2 - 2n\} = \frac{n^2}{2}(n^2+2n-1)\end{aligned}$$

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Series
Exercise F, Question 12

Question:

a Show that $\sum_{r=1}^{n} r(r+1) = \frac{n}{3}(n+1)(n+2)$.

b Hence calculate $\sum_{r=31}^{60} r(r+1)$.

Solution:

a

$$\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r = \frac{n}{6}(n+1)(2n+1) + \frac{n}{2}(n+1) \quad = \frac{n}{6}(n+1)\{2n+1+3\}$$

$$= \frac{n}{3}(n+1)(n+2)$$

b Substitute $n = 60$ and $n = 30$ into the result for part (a), and subtract. The result = 65720.

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Series
Exercise F, Question 13

Question:

a Show that $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{n}{4}(n+1)(n+2)(n+3)$.

b Hence evaluate $3\times4\times5+4\times5\times6+5\times6\times7+\ldots+40\times41\times42$.

Solution:

a

$$\begin{aligned}\sum_{r=1}^{n} r^3+3\sum_{r=1}^{n} r^2+2\sum_{r=1}^{n} r &= \frac{n^2}{4}(n+1)^2+\frac{n}{2}(n+1)(2n+1)+n(n+1)\\ &= \frac{n}{4}(n+1)\{(n(n+1)+2(2n+1)+4\}\\ &= \frac{n}{4}(n+1)(n+2)(n+3)\end{aligned}$$

b $3\times4\times5+4\times5\times6+5\times6\times7+\ldots+40\times41\times42 = \sum_{r=3}^{40} r(r+1)(r+2)$

$$\begin{aligned}\sum_{r=3}^{40} r(r+1)(r+2) &= \sum_{r=1}^{40} r(r+1)(r+2)-\sum_{r=1}^{2} r(r+1)(r+2)\\ &= \frac{40\times41\times42\times43}{4}-\frac{2\times3\times4\times5}{4} = 740430\end{aligned}$$

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Series
Exercise F, Question 14

Question:

a Show that $\sum_{r=1}^{n} r\{2(n-r)+1\} = \frac{n}{6}(n+1)(2n+1)$.

b Hence sum the series $(2n-1)+2(2n-3)+3(2n-5)+\ldots+n$

Solution:

a Series can be written as $(2n+1)\sum_{r=1}^{n} r - 2\sum_{r=1}^{n} r^2$ as n is a constant.

$$= (2n+1)\frac{n}{2}(n+1) - \frac{n}{3}(n+1)(2n+1)$$
$$= \frac{n}{6}(n+1)(2n+1)$$

b $\sum_{r=1}^{n} r[2(n-r)+1] = (2n-1)+2[(2n-4)+1]+3[(2n-6)+1]+\ldots+n[2(n-n)+1]$

$= (2n-1)+2(2n+3)+3(2n+5)+\ldots+n$, the series in part (b).

The sum, therefore, is $\frac{n}{6}(n+1)(2n+1)$

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Series
Exercise F, Question 15

Question:

a Show that when n is even,

$$1^3 - 2^3 + 3^3 - \ldots - n^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 - 16\left(1^3 + 2^3 + 3^3 + \ldots + \left(\frac{n}{2}\right)^3\right)$$

$$= \sum_{r=1}^{n} r^3 - 16\sum_{r=1}^{\frac{n}{2}} r^3.$$

b Hence show that, for n even, $1^3 - 2^3 + 3^3 - \ldots - n^3 = -\frac{n^2}{4}(2n+3)$

c Deduce the sum of $1^3 - 2^3 + 3^3 - \ldots - 40^3$.

Solution:

a

$$\begin{aligned}
1^3 - 2^3 + 3^3 - \ldots - n^3 &= (1^3 + 2^3 + 3^3 + \ldots + n^3) - 2(2^3 + 4^3 + 6^3 + \ldots + n^3) \\
&= (1^3 + 2^3 + 3^3 + \ldots + n^3) - 2\left\{2^3(1^3 + 2^3 + 3^3 + \ldots + \left(\frac{n}{2}\right)^3\right\} \text{ as } n \text{ is even} \\
&= (1^3 + 2^3 + 3^3 + \ldots + n^3) - 16\left\{1^3 + 2^3 + 3^3 + \ldots + \left(\frac{n}{2}\right)^3\right\} \\
&= \sum_{r=1}^{n} r^3 - 16\sum_{r=1}^{\frac{n}{2}} r^3 \text{ [As } n \text{ is even, } \frac{n}{2} \text{ is an integer]}
\end{aligned}$$

b

$$\begin{aligned}
\sum_{r=1}^{n} r^3 - 16\sum_{r=1}^{\frac{n}{2}} r^3 &= \frac{n^2}{4}(n+1)^2 - 16\frac{\left(\frac{n}{2}\right)^2\left(\frac{n}{2}+1\right)^2}{4} \\
&= \frac{n^2}{4}(n+1)^2 - 4\frac{n^2}{4}\frac{(n+2)^2}{4} \\
&= \frac{n^2}{4}(n+1)^2 - \frac{n^2}{4}(n+2)^2 \\
&= \frac{n^2}{4}\{(n+1)^2 - (n+2)^2\} \\
&= \frac{n^2}{4}(-2n-3) = -\frac{n^2}{4}(2n+3)
\end{aligned}$$

c Substituting $n = 40$, gives -33200.

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Proof by mathematical induction
Exercise A, Question 1

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

Solution:

$$n = 1; \text{LHS} \;=\; \sum_{r=1}^{1} r = 1$$
$$\text{RHS} \;=\; \frac{1}{2}(1)(2) = 1$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$.

With $n = k+1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} r &= 1+2+3+ \geq +k+(k+1)\\ &= \frac{1}{2}k(k+1)+(k+1)\\ &= \frac{1}{2}(k+1)(k+2)\\ &= \frac{1}{2}(k+1)(k+1+1)\end{aligned}$$

Therefore, summation formula is true when $n = k+1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k+1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 2

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1} r^3 = 1$$
$$\text{RHS} = \frac{1}{4}(1)^2(2)^2 = \frac{1}{4}(4) = 1$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$.

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}
\sum_{r=1}^{k+1} r^3 &= 1^3 + 2^3 + 3^3 + \geq + k^3 + (k+1)^3 \\
&= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\
&= \frac{1}{4}(k+1)^2\left[k^2 + 4(k+1)\right] \\
&= \frac{1}{4}(k+1)^2(k^2 + 4k + 4) \\
&= \frac{1}{4}(k+1)^2(k+2)^2 \\
&= \frac{1}{4}(k+1)^2(k+1+1)^2
\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 3

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} r(r-1) = \frac{1}{3}n(n+1)(n-1)$$

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1} r(r-1) = 1(0) = 0$$
$$\text{RHS} = \frac{1}{3}(1)(2)(0) = 0$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} r(r-1) = \frac{1}{3}k(k+1)(k-1)$.

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} r(r-1) &= 1(0) + 2(1) + 3(2) + \geq + k(k-1) + (k+1)k \\ &= \frac{1}{3}k(k+1)(k-1) + (k+1)k \\ &= \frac{1}{3}k(k+1)[(k-1) + 3] \\ &= \frac{1}{3}k(k+1)(k+2) \\ &= \frac{1}{3}(k+1)(k+2)k \\ &= \frac{1}{3}(k+1)(k+1+1)(k+1-1)\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 4

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$(1\times 6)+(2\times 7)+(3\times 8)+ \geq +n(n+5)=\frac{1}{3}n(n+1)(n+8)$$

Solution:

The identity $(1\times 6)+(2\times 7)+(3\times 8)+ \geq +n(n+5)=\frac{1}{3}n(n+1)(n+8)$ can be rewritten as $\sum_{r=1}^{n} r(r+5)=\frac{1}{3}n(n+1)(n+8)$.

$$n=1;\ \text{LHS} = \sum_{r=1}^{1} r(r+5)=1(6)=6$$
$$\text{RHS} = \frac{1}{3}(1)(2)(9)=\frac{1}{3}(18)=6$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} r(r+5)=\frac{1}{3}k(k+1)(k+8)$.

With $n = k+1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} r(r+5) &= 1(6)+2(7)+3(8)+ \geq +k(k+5)+(k+1)(k+6)\\
&= \frac{1}{3}k(k+1)(k+8)+(k+1)(k+6)\\
&= \frac{1}{3}(k+1)[k(k+8)+3(k+6)]\\
&= \frac{1}{3}(k+1)\left[k^2+8k+3k+18\right]\\
&= \frac{1}{3}(k+1)\left[k^2+11k+18\right]\\
&= \frac{1}{3}(k+1)(k+9)(k+2)\\
&= \frac{1}{3}(k+1)(k+2)(k+9)\\
&= \frac{1}{3}(k+1)(k+1+1)(k+1+8)\end{aligned}$$

Therefore, summation formula is true when $n = k+1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k+1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 5

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} r(3r-1) = n^2(n+1)$$

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1} r(3r-1) = 1(2) = 2$$
$$\text{RHS} = 1^2(2) = (1)(2) = 2$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} r(3r-1) = k^2(k+1)$.

With $n = k+1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} r(3r-1) &= 1(2) + 2(5) + 3(8) + \geq + k(3k-1) + (k+1)(3(k+1)-1)\\ &= k^2(k+1) + (k+1)(3k+3-1)\\ &= k^2(k+1) + (k+1)(3k+2)\\ &= (k+1)\left[k^2 + 3k + 2\right]\\ &= (k+1)(k+2)(k+1)\\ &= (k+1)^2(k+2)\\ &= (k+1)^2(k+1+1)\end{aligned}$$

Therefore, summation formula is true when $n = k+1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k+1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 6

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(4n^2-1)$$

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1}(2r-1)^2 = 1^2 = 1$$

$$\text{RHS} = \frac{1}{3}(1)(4-1) = \frac{1}{3}(1)(3) = 1$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k}(2r-1)^2 = \frac{1}{3}k(4k^2-1) = \frac{1}{3}k(2k+1)(2k-1)$.

With $n = k+1$ terms the summation formula becomes:

$$\begin{aligned}
\sum_{r=1}^{k+1}(2r-1)^2 &= 1^2 + 3^2 + 5^2 + \geq + (2k-1)^2 + (2(k+1)-1)^2 \\
&= \frac{1}{3}k(4k^2-1) + (2k+2-1)^2 \\
&= \frac{1}{3}k(4k^2-1) + (2k+1)^2 \\
&= \frac{1}{3}k(2k+1)(2k-1) + (2k+1)^2 \\
&= \frac{1}{3}(2k+1)[k(2k-1) + 3(2k+1)] \\
&= \frac{1}{3}(2k+1)\left[2k^2 - k + 6k + 3\right] \\
&= \frac{1}{3}(2k+1)\left[2k^2 + 5k + 3\right] \\
&= \frac{1}{3}(2k+1)(k+1)(2k+3) \\
&= \frac{1}{3}(k+1)(2k+3)(2k+1) \\
&= \frac{1}{3}(k+1)[2(k+1)+1][2(k+1)-1] \\
&= \frac{1}{3}(k+1)\left[4(k+1)^2 - 1\right]
\end{aligned}$$

Therefore, summation formula is true when $n = k+1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k+1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 7

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} 2^r = 2^{n+1} - 2$$

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1} 2^r = 2^1 = 2$$
$$\text{RHS} = 2^2 - 2 = 4 - 2 = 2$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} 2^r = 2^{k+1} - 2.$

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} 2^r &= 2^1 + 2^2 + 2^3 + \geq + 2^k + 2^{k+1} \\ &= 2^{k+1} - 2 + 2^{k+1} \\ &= 2(2^{k+1}) - 2 \\ &= 2^1(2^{k+1}) - 2 \\ &= 2^{1+k+1} - 2 \\ &= 2^{k+1+1} - 2\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 8

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} 4^{r-1} = \frac{4^n - 1}{3}$$

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1} 4^{r-1} = 4^0 = 1$$

$$\text{RHS} = \frac{4-1}{3} = \frac{3}{3} = 1$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} 4^{r-1} = \frac{4^k - 1}{3}$.

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}
\sum_{r=1}^{k+1} 4^{r-1} &= 4^0 + 4^1 + 4^2 + \geq + 4^{k-1} + 4^{k+1-1} \\
&= \frac{4^k - 1}{3} + 4^k \\
&= \frac{4^k - 1}{3} + \frac{3(4^k)}{3} \\
&= \frac{4^k - 1 + 3(4^k)}{3} \\
&= \frac{4(4^k) - 1}{3} \\
&= \frac{4^1(4^k) - 1}{3} \\
&= \frac{4^{k+1} - 1}{3}
\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 9

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} r(r\,!) = (n+1)\,! - 1$$

Solution:

$$\begin{aligned} n = 1;\ \text{LHS} &= \sum_{r=1}^{1} r(r\,!) = 1(1\,!) = 1(1) = 1 \\ \text{RHS} &= 2\,! - 1 = 2 - 1 = 1 \end{aligned}$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} r(r\,!) = (k+1)\,! - 1$.

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned} \sum_{r=1}^{k+1} r(r\,!) &= 1(1\,!) + 2(2\,!) + 3(3\,!) + \geq + k(k\,!) + (k+1)[(k+1)\,!] \\ &= (k+1)\,! - 1 + (k+1)[(k+1)\,!] \\ &= (k+1)\,! + (k+1)[(k+1)\,!] - 1 \\ &= (k+1)\,!\,[1 + k + 1] - 1 \\ &= (k+1)\,!\,(k+2) - 1 \\ &= (k+2)\,! - 1 \\ &= (k+1+1)\,! - 1 \end{aligned}$$

Therefore, summation formula is true when $n = k + 1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 10

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{2n} r^2 = \frac{1}{3}n(2n+1)(4n+1)$$

Solution:

$$n = 1; \text{LHS} = \sum_{r=1}^{2} r^2 = 1^2 + 2^2 = 1 + 4 = 5$$

$$\text{RHS} = \frac{1}{3}(1)(3)(5) = \frac{1}{3}(15) = 5$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{2k} r^2 = \frac{1}{3}k(2k+1)(4k+1)$.

With $n = k+1$ terms the summation formula becomes:

$$\begin{aligned}
\sum_{r=1}^{2(k+1)} r^2 = \sum_{r=1}^{2k+2} r^2 &= 1^2 + 2^2 + 3^2 + \geq + k^2 + (2k+1)^2 + (2k+2)^2 \\
&= \frac{1}{3}k(2k+1)(4k+1) + (2k+1)^2 + (2k+2)^2 \\
&= \frac{1}{3}k(2k+1)(4k+1) + (2k+1)^2 + 4(k+1)^2 \\
&= \frac{1}{3}(2k+1)[k(4k+1) + 3(2k+1)] + 4(k+1)^2 \\
&= \frac{1}{3}(2k+1)[4k^2 + 7k + 3] + 4(k+1)^2 \\
&= \frac{1}{3}(2k+1)(4k+3)(k+1) + 4(k+1)^2 \\
&= \frac{1}{3}(k+1)[(2k+1)(4k+3) + 12(k+1)] \\
&= \frac{1}{3}(k+1)[8k^2 + 6k + 4k + 3 + 12k + 12] \\
&= \frac{1}{3}(k+1)[8k^2 + 22k + 15] \\
&= \frac{1}{3}(k+1)(2k+3)(4k+5) \\
&= \frac{1}{3}(k+1)[2(k+1)+1][4(k+1)+1]
\end{aligned}$$

Therefore, summation formula is true when $n = k+1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k+1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
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Proof by mathematical induction
Exercise B, Question 1

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$8^n - 1$ is divisible by 7

Solution:

Let $\text{f}(n) = 8^n - 1$, where $n \in \mathbb{Z}^+$.

$\therefore$ $\text{f}(1) = 8^1 - 1 = 7$, which is divisible by 7.

$\therefore$ $\text{f}(n)$ is divisible by 7 when $n = 1$.

Assume that for $n = k$,

$\text{f}(k) = 8^k - 1$ is divisible by 7 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ \text{f}(k+1) &= 8^{k+1} - 1\\ &= 8^k.8^1 - 1\\ &= 8(8^k) - 1\end{aligned}$$

$$\begin{aligned}\therefore\ \text{f}(k+1) - \text{f}(k) &= [8(8^k) - 1] - [8^k - 1]\\ &= 8(8^k) - 1 - 8^k + 1\\ &= 7(8^k)\end{aligned}$$

$\therefore$ $\text{f}(k+1) = \text{f}(k) + 7(8^k)$

As both $\text{f}(k)$ and $7(8^k)$ are divisible by 7 then the sum of these two terms must also be divisible by 7. Therefore $\text{f}(n)$ is divisible by 7 when $n = k + 1$.

If $\text{f}(n)$ is divisible by 7 when $n = k$, then it has been shown that $\text{f}(n)$ is also divisible by 7 when $n = k + 1$. As $\text{f}(n)$ is divisible by 7 when $n = 1$, $\text{f}(n)$ is also divisible by 7 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 2

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$3^{2n} - 1$ is divisible by 8

Solution:

Let $f(n) = 3^{2n} - 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 3^{2(1)} - 1 = 9 - 1 = 8$, which is divisible by 8.

$\therefore\ f(n)$ is divisible by 8 when $n = 1$.

Assume that for $n = k$,

$f(k) = 3^{2k} - 1$ is divisible by 8 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ f(k+1) &= 3^{2(k+1)} - 1 \\ &= 3^{2k+2} - 1 \\ &= 3^{2k}.3^2 - 1 \\ &= 9(3^{2k}) - 1\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1) - f(k) &= [9(3^{2k}) - 1] - [3^{2k} - 1] \\ &= 9(3^{2k}) - 1 - 3^{2k} + 1 \\ &= 8(3^{2k})\end{aligned}$$

$$\therefore\ f(k+1) = f(k) + 8(3^{2k})$$

As both $f(k)$ and $8(3^{2k})$ are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore $f(n)$ is divisible by 8 when $n = k + 1$.

If $f(n)$ is divisible by 8 when $n = k$, then it has been shown that $f(n)$ is also divisible by 8 when $n = k + 1$. As $f(n)$ is divisible by 8 when $n = 1$, $f(n)$ is also divisible by 8 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 3

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$5^n + 9^n + 2$ is divisible by 4

Solution:

Let $f(n) = 5^n + 9^n + 2$, where $n \in \mathbb{Z}^+$.

$\therefore$ $f(1) = 5^1 + 9^1 + 2 = 5 + 9 + 2 = 16$, which is divisible by 4.

$\therefore$ $f(n)$ is divisible by 4 when $n = 1$.

Assume that for $n = k$,

$f(k) = 5^k + 9^k + 2$ is divisible by 4 for $k \in \mathbb{Z}^+$.

$$\begin{aligned} \therefore\ f(k+1) &= 5^{k+1} + 9^{k+1} + 2 \\ &= 5^k.5^1 + 9^k.9^1 + 2 \\ &= 5(5^k) + 9(9^k) + 2 \end{aligned}$$

$$\begin{aligned} \therefore\ f(k+1) - f(k) &= [5(5^k) + 9(9^k) + 2] - [5^k + 9^k + 2] \\ &= 5(5^k) + 9(9^k) + 2 - 5^k - 9^k - 2 \\ &= 4(5^k) + 8(9^k) \\ &= 4[5^k + 2(9)^k] \end{aligned}$$

$\therefore$ $f(k+1) = f(k) + 4[5^k + 2(9)^k]$

As both $f(k)$ and $4[5^k + 2(9)^k]$ are divisible by 4 then the sum of these two terms must also be divisible by 4. Therefore f (n) is divisible by 4 when $n = k + 1$.

If $f(n)$ is divisible by 4 when $n = k$, then it has been shown that $f(n)$ is also divisible by 4 when $n = k + 1$. As $f(n)$ is divisible by 4 when $n = 1$, $f(n)$ is also divisible by 4 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 4

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$2^{4n} - 1$ is divisible by 15

Solution:

Let $f(n) = 2^{4n} - 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 2^{4(1)} - 1 = 16 - 1 = 15$, which is divisible by 15.

$\therefore\ f(n)$ is divisible by 15 when $n = 1$.

Assume that for $n = k$,

$f(k) = 2^{4k} - 1$ is divisible by 15 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ f(k+1) &= 2^{4(k+1)} - 1\\ &= 2^{4k+4} - 1\\ &= 2^{4k}.2^4 - 1\\ &= 16(2^{4k}) - 1\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1) - f(k) &= [16(2^{4k}) - 1] - [2^{4k} - 1]\\ &= 16(2^{4k}) - 1 - 2^{4k} + 1\\ &= 15(8^k)\end{aligned}$$

$\therefore\ f(k+1) = f(k) + 15(8^k)$

As both $f(k)$ and $15(8^k)$ are divisible by 15 then the sum of these two terms must also be divisible by 15. Therefore $f(n)$ is divisible by 15 when $n = k + 1$.

If $f(n)$ is divisible by 15 when $n = k$, then it has been shown that $f(n)$ is also divisible by 15 when $n = k + 1$. As $f(n)$ is divisible by 15 when $n = 1$, $f(n)$ is also divisible by 15 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 5

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$3^{2n-1} + 1$ is divisible by 4

Solution:

Let $f(n) = 3^{2n-1} + 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 3^{2(1)-1} + 1 = 3 + 1 = 4$, which is divisible by 4.

$\therefore\ f(n)$ is divisible by 4 when $n = 1$.

Assume that for $n = k$,

$f(k) = 3^{2k-1} + 1$ is divisible by 4 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ f(k+1) &= 3^{2(k+1)-1} + 1 \\ &= 3^{2k+2-1} + 1 \\ &= 3^{2k-1}.3^2 + 1 \\ &= 9\left(3^{2k-1}\right) + 1\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1) - f(k) &= \left[9(3^{2k-1}) + 1\right] - \left[3^{2k-1} + 1\right] \\ &= 9\left(3^{2k-1}\right) + 1 - 3^{2k-1} - 1 \\ &= 8\left(3^{2k-1}\right)\end{aligned}$$

$$\therefore\ f(k+1) = f(k) + 8\left(3^{2k-1}\right)$$

As both $f(k)$ and $8\left(3^{2k-1}\right)$ are divisible by 4 then the sum of these two terms must also be divisible by 4. Therefore $f(n)$ is divisible by 4 when $n = k + 1$.

If $f(n)$ is divisible by 4 when $n = k$, then it has been shown that $f(n)$ is also divisible by 4 when $n = k + 1$. As $f(n)$ is divisible by 4 when $n = 1$, $f(n)$ is also divisible by 8 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise B, Question 6

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$n^3 + 6n^2 + 8n$ is divisible by 3

Solution:

Let $f(n) = n^3 + 6n^2 + 8n$, where $n \geq 1$ and $n \in \mathbb{Z}^+$.

$\therefore$ $f(1) = 1 + 6 + 8 = 15$, which is divisible by 3.

$\therefore$ $f(n)$ is divisible by 3 when $n = 1$.

Assume that for $n = k$,

$f(k) = k^3 + 6k^2 + 8k$ is divisible by 3 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}
\therefore\ f(k+1) &= (k+1)^3 + 6(k+1)^2 + 8(k+1) \\
&= k^3 + 3k^2 + 3k + 1 + 6(k^2 + 2k + 1) + 8(k+1) \\
&= k^3 + 3k^2 + 3k + 1 + 6k^2 + 12k + 6 + 8k + 8 \\
&= k^3 + 9k^2 + 23k + 15
\end{aligned}$$

$$\begin{aligned}
\therefore\ f(k+1) - f(k) &= [k^3 + 9k^2 + 23k + 15] - [k^3 + 6k^2 + 8k] \\
&= 3k^2 + 15k + 15 \\
&= 3(k^2 + 5k + 5)
\end{aligned}$$

$\therefore$ $f(k+1) = f(k) + 3\left(k^2 + 5k + 5\right)$

As both $f(k)$ and $3\left(k^2 + 5k + 5\right)$ are divisible by 3 then the sum of these two terms must also be divisible by 3.

Therefore $f(n)$ is divisible by 3 when $n = k + 1$.

If $f(n)$ is divisible by 3 when $n = k$, then it has been shown that $f(n)$ is also divisible by 3 when $n = k + 1$. As $f(n)$ is divisible by 3 when $n = 1$, $f(n)$ is also divisible by 3 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise B, Question 7

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$n^3 + 5n$ is divisible by 6

Solution:

Let $f(n) = n^3 + 5n$, where $n \geq 1$ and $n \in \mathbb{Z}^+$.

$\therefore$ $f(1) = 1 + 5 = 6$, which is divisible by 6.

$\therefore$ $f(n)$ is divisible by 6 when $n = 1$.

Assume that for $n = k$,

$f(k) = k^3 + 5k$ is divisible by 6 for $k \in \mathbb{Z}^+$.

$$\begin{aligned} \therefore\ f(k+1) &= (k+1)^3 + 5(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 5(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 5k + 5 \\ &= k^3 + 3k^2 + 8k + 6 \end{aligned}$$

$$\begin{aligned} \therefore\ f(k+1) - f(k) &= [k^3 + 3k^2 + 8k + 6] - [k^3 + 5k] \\ &= 3k^2 + 3k + 6 \\ &= 3k(k+1) + 6 \\ &= 3(2m) + 6 \\ &= 6m + 6 \\ &= 6(m+1) \end{aligned}$$

Let $k(k+1) = 2m, m \in \mathbb{Z}^+$, as the product of two consecutive integers must be even.

$\therefore$ $f(k+1) = f(k) + 6(m+1)$.

As both $f(k)$ and $6(m+1)$ are divisible by 6 then the sum of these two terms must also be divisible by 6. Therefore $f(n)$ is divisible by 6 when $n = k + 1$.

If $f(n)$ is divisible by 6 when $n = k$, then it has been shown that $f(n)$ is also divisible by 6 when $n = k + 1$. As $f(n)$ is divisible by 6 when $n = 1$, $f(n)$ is also divisible by 6 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction

Exercise B, Question 8

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$2^n.3^{2n} - 1$ is divisible by 17

Solution:

Let $f(n) = 2^n.3^{2n} - 1$, where $n \in \mathbb{Z}^+$.

$\therefore$ $f(1) = 2^1.3^{2(1)} - 1 = 2(9) - 1 = 18 - 1 = 17$, which is divisible by 17.

$\therefore$ $f(n)$ is divisible by 17 when $n = 1$.

Assume that for $n = k$,

$f(k) = 2^k.3^{2k} - 1$ is divisible by 17 for $k \in \mathbb{Z}^+$.

$$\begin{aligned} \therefore\ f(k+1) &= 2^{k+1}.3^{2(k+1)} - 1 \\ &= 2^k(2)^1(3)^{2k}(3)^2 - 1 \\ &= 2^k(2)^1(3)^{2k}(9) - 1 \\ &= 18(2^k.3^{2k}) - 1 \end{aligned}$$

$$\begin{aligned} \therefore\ f(k+1) - f(k) &= \left[18(2^k.3^{2k}) - 1\right] - \left[2^k.3^{2k} - 1\right] \\ &= 18(2^k.3^{2k}) - 1 - 2^k.3^{2k} + 1 \\ &= 17(2^k.3^{2k}) \end{aligned}$$

$\therefore$ $f(k+1) = f(k) + 17(2^k.3^{2k})$

As both $f(k)$ and $17(2^k.3^{2k})$ are divisible by 17 then the sum of these two terms must also be divisible by 17.

Therefore $f(n)$ is divisible by 17 when $n = k + 1$.

If $f(n)$ is divisible by 17 when $n = k$, then it has been shown that $f(n)$ is also divisible by 17 when $n = k + 1$. As $f(n)$ is divisible by 17 when $n = 1$, $f(n)$ is also divisible by 17 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise B, Question 9

Question:

$f(n) = 13^n - 6^n$, $n \in \mathbb{Z}^+$.

a Express for $k \in \mathbb{Z}^+$, $f(k+1) - 6f(k)$ in terms of k, simplifying your answer.

b Use the method of mathematical induction to prove that $f(n)$ is divisible by 7 for all $n \in \mathbb{Z}^+$.

Solution:

a

$$\begin{aligned} f(k+1) &= 13^{k+1} - 6^{k+1} \\ &= 13^k.13^1 - 6^k.6^1 \\ &= 13(13^k) - 6(6^k) \end{aligned}$$

$$\begin{aligned} \therefore\ f(k+1) - 6f(k) &= \left[13(13^k) - 6(6^k)\right] - 6\left[13^k - 6^k\right] \\ &= 13(13^k) - 6(6^k) - 6(13^k) + 6(6^k) \\ &= 7(13^k) \end{aligned}$$

b $f(n) = 13^n - 6^n$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 13^1 - 6^1 = 7$, which is divisible by 7.

$\therefore\ f(n)$ is divisible by 7 when $n = 1$.

Assume that for $n = k$,

$f(k) = 13^k - 6^k$ is divisible by 7 for $k \in \mathbb{Z}^+$.

From (a), $f(k+1) = 6f(k) + 7(13^k)$

As both $6f(k)$ and $7(13^k)$ are divisible by 7 then the sum of these two terms must also be divisible by 7. Therefore $f(n)$ is divisible by 7 when $n = k+1$.

If $f(n)$ is divisible by 7 when $n = k$, then it has been shown that $f(n)$ is also divisible by 7 when $n = k+1$. As $f(n)$ is divisible by 7 when $n = 1$, $f(n)$ is also divisible by 7 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 10

Question:

$g(n) = 5^{2n} - 6n + 8$, $n \in \mathbb{Z}^+$.

a Express for $k \in \mathbb{Z}^+$, $g(k+1) - 25g(k)$ in terms of k, simplifying your answer.

b Use the method of mathematical induction to prove that $g(n)$ is divisible by 9 for all $n \in \mathbb{Z}^+$.

Solution:

a

$$\begin{aligned} g(k+1) &= 5^{2(k+1)} - 6(k+1) + 8 \\ &= 5^{2k}.5^2 - 6k - 6 + 8 \\ &= 25(5^{2k}) - 6k + 2 \end{aligned}$$

$$\begin{aligned} \therefore\ g(k+1) - 25g(k) &= \left[25(5^{2k}) - 6k + 2\right] - 25\left[5^{2k} - 6k + 8\right] \\ &= 25(5^{2k}) - 6k + 2 - 25(5^{2k}) + 150k - 200 \\ &= 144k - 198 \end{aligned}$$

b

$g(n) = 5^{2n} - 6n + 8$, where $n \in \mathbb{Z}^+$.

$\therefore\ g(1) = 5^2 - 6(1) + 8 = 25 - 6 + 8 = 27$, which is divisible by 9.

$\therefore\ g(n)$ is divisible by 9 when $n = 1$.

Assume that for $n = k$,

$g(k) = 5^{2k} - 6k + 8$ is divisible by 9 for $k \in \mathbb{Z}^+$.

$$\begin{aligned} \text{From(a), } g(k+1) &= 25g(k) + 144n - 198 \\ &= 25g(k) + 18(8n - 11) \end{aligned}$$

As both $25g(k)$ and $18(8n - 11)$ are divisible by 9 then the sum of these two terms must also be divisible by 9. Therefore $g(n)$ is divisible by 9 when $n = k + 1$.

If $g(n)$ is divisible by 9 when $n = k$, then it has been shown that $g(n)$ is also divisible by 9 when $n = k + 1$. As $g(n)$ is divisible by 9 when $n = 1$, $g(n)$ is also divisible by 9 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 11

Question:

Use the method of mathematical induction to prove that $8^n - 3^n$ is divisible by 5 for all $n \in \mathbb{Z}^+$.

Solution:

$f(n) = 8^n - 3^n$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 8^1 - 3^1 = 5$, which is divisible by 5.

$\therefore\ f(n)$ is divisible by 5 when $n = 1$.

Assume that for $n = k$,

$f(k) = 8^k - 3^k$ is divisible by 5 for $k \in \mathbb{Z}^+$.

$$\begin{aligned} \therefore\ f(k+1) &= 8^{k+1} - 3^{k+1} \\ &= 8^k.8^1 - 3^k.3^1 \\ &= 8(8^k) - 3(3^k) \end{aligned}$$

$$\begin{aligned} \therefore\ f(k+1) - 3f(k) &= \left[8(8^k) - 3(3^k)\right] - 3\left[8^k - 3^k\right] \\ &= 8(8^k) - 3(3^k) - 3(8^k) + 3(3^k) \\ &= 5(8^k) \end{aligned}$$

From (a), $f(k+1) = f(k) + 5(8^k)$

As both $f(k)$ and $5(8^k)$ are divisible by 5 then the sum of these two terms must also be divisible by 5. Therefore $f(n)$ is divisible by 5 when $n = k + 1$.

If $f(n)$ is divisible by 5 when $n = k$, then it has been shown that $f(n)$ is also divisible by 5 when $n = k + 1$. As $f(n)$ is divisible by 5 when $n = 1$, $f(n)$ is also divisible by 5 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 12

Question:

Use the method of mathematical induction to prove that $3^{2n+2}+8n-9$ is divisible by 8 for all $n \in \mathbb{Z}^+$.

Solution:

$f(n) = 3^{2n+2}+8n-9$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) \quad = 3^{2(1)+2}+8(1)-9$

$= 3^4+8-9 = 81-1 = 80$, which is divisible by 8.

$\therefore\ f(n)$ is divisible by 8 when $n=1$.

Assume that for $n = k$,

$f(k) = 3^{2k+2}+8k-9$ is divisible by 8 for $k \in \mathbb{Z}^+$.

$$\begin{aligned} f(k+1) &= 3^{2(k+1)+2}+8(k+1)-9 \\ &= 3^{2k+2+2}+8(k+1)-9 \\ &= 3^{2k+2}.(3^2)+8k+8-9 \\ &= 9(3^{2k+2})+8k-1 \end{aligned}$$

$$\begin{aligned} \therefore\ f(k+1)-f(k) &= \left[9(3^{2k+2})+8k-1\right]-\left[3^{2k+2}+8k-9\right] \\ &= 9(3^{2k+2})+8k-1-3^{2k+2}-8k+9 \\ &= 8(3^{2k+2})+8 \\ &= 8\left[3^{2k+2}+1\right] \end{aligned}$$

$$\therefore\ f(k+1) = f(k)+8\left[3^{2k+2}+1\right]$$

As both $f(k)$ and $8\left[3^{2k+2}+1\right]$ are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore $f(n)$ is divisible by 8 when $n=k+1$.

If $f(n)$ is divisible by 8 when $n=k$, then it has been shown that $f(n)$ is also divisible by 8 when $n=k+1$. As $f(n)$ is divisible by 8 when $n=1$, $f(n)$ is also divisible by 8 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 13

Question:

Use the method of mathematical induction to prove that $2^{6n} + 3^{2n-2}$ is divisible by 5 for all $n \in \mathbb{Z}^+$.

Solution:

$f(n) = 2^{6n} + 3^{2n-2}$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 2^{6(1)} + 3^{2(1)-2} = 2^6 + 3^0 = 64 + 1 = 65$, which is divisible by 5.

$\therefore\ f(n)$ is divisible by 5 when $n = 1$.

Assume that for $n = k$,

$f(k) = 2^{6k} + 3^{2k-2}$ is divisible by 5 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}
\therefore\ f(k+1) &= 2^{6(k+1)} + 3^{2(k+1)-2} \\
&= 2^{6k+6} + 3^{2k+2-2} \\
&= 2^6\left(2^{6k}\right) + 3^2\left(3^{2k-2}\right) \\
&= 64\left(2^{6k}\right) + 9\left(3^{2k-2}\right)
\end{aligned}$$

$$\begin{aligned}
\therefore\ f(k+1) - f(k) &= \left[64\left(2^{6k}\right) + 9\left(3^{2k-2}\right)\right] - \left[2^{6k} + 3^{2k-2}\right] \\
&= 64\left(2^{6k}\right) + 9\left(3^{2k-2}\right) - 2^{6k} - 3^{2k-2} \\
&= 63\left(2^{6k}\right) + 8\left(3^{2k-2}\right) \\
&= 63\left(2^{6k}\right) + 63\left(3^{2k-2}\right) - 55\left(3^{2k-2}\right) \\
&= 63\left[2^{6k} + 3^{2k-2}\right] - 55\left(3^{2k-2}\right)
\end{aligned}$$

$$\begin{aligned}
\therefore\ f(k+1) &= f(k) + 63\left[2^{6k} + 3^{2k-2}\right] - 55\left(3^{2k-2}\right) \\
&= f(k) + 63f(k) - 55\left(3^{2k-2}\right) \\
&= 64f(k) - 55\left(3^{2k-2}\right) \\
\therefore\ f(k+1) &= 64f(k) - 55\left(3^{2k-2}\right)
\end{aligned}$$

As both $64f(k)$ and $-55\left(3^{2k-2}\right)$ are divisible by 5 then the sum of these two terms must also be divisible by 5. Therefore $f(n)$ is divisible by 5 when $n = k + 1$.

If $f(n)$ is divisible by 5 when $n = k$, then it has been shown that $f(n)$ is also divisible by 5 when $n = k + 1$. As $f(n)$ is divisible by 5 when $n = 1$, $f(n)$ is also divisible by 5 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise C, Question 1

Question:

Given that $u_{n+1} = 5u_n + 4$, $u_1 = 4$, prove by induction that $u_n = 5^n - 1$.

Solution:

$n = 1$; $u_1 = 5^1 - 1 = 4$, as given.

$n = 2$; $u_2 = 5^2 - 1 = 24$, from the general statement.

and $u_2 = 5u_1 + 4 = 5(4) + 4 = 24$, from the recurrence relation.

So u_n is true when $n = 1$ and also true when $n = 2$.

Assume that for $n = k$ that, $u_k = 5^k - 1$ is true for $k \in \mathbb{Z}^+$.

$$\begin{aligned} \text{Then } u_{k+1} &= 5u_k + 4 \\ &= 5\left(5^k - 1\right) + 4 \\ &= 5^{k+1} - 5 + 4 \\ &= 5^{k+1} - 1 \end{aligned}$$

Therefore, the general statement, $u_n = 5^n - 1$ is true when $n = k + 1$.

If u_n is true when $n = k$, then it has been shown that $u_n = 5^n - 1$ is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise C, Question 2

Question:

Given that $u_{n+1} = 2u_n + 5$, $u_1 = 3$, prove by induction that $u_n = 2^{n+2} - 5$.

Solution:

$n = 1$; $u_1 = 2^{1+2} - 5 = 8 - 5 = 3$, as given.

$n = 2$; $u_2 = 2^4 - 5 = 16 - 5 = 11$, from the general statement.

and $u_2 = 2u_1 + 5 = 2(3) + 5 = 11$, from the recurrence relation.

So u_n is true when $n = 1$ and also true when $n = 2$.

Assume that for $n = k$ that, $u_k = 2^{k+2} - 5$ is true for $k \in \mathbb{Z}^+$.

$$\begin{aligned}
\text{Then } u_{k+1} &= 2u_k + 5 \\
&= 2\left(2^{k+2} - 5\right) + 5 \\
&= 2^{k+3} - 10 + 5 \\
&= 2^{k+1+2} - 5
\end{aligned}$$

Therefore, the general statement, $u_n = 2^{n+2} - 5$ is true when $n = k + 1$.

If u_n is true when $n = k$, then it has been shown that $u_n = 2^{n+2} - 5$ is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise C, Question 3

Question:

Given that $u_{n+1} = 5u_n - 8,\ u_1 = 3$, prove by induction that $u_n = 5^{n-1} + 2$.

Solution:

$n = 1$; $u_1 = 5^{1-1} + 2 = 1 + 2 = 3$, as given.

$n = 2$; $u_2 = 5^{2-1} + 2 = 5 + 2 = 7$, from the general statement.

and $u_2 = 5u_1 - 8 = 5(3) - 8 = 7$, from the recurrence relation.

So u_n is true when $n = 1$ and also true when $n = 2$.

Assume that for $n = k$ that, $u_k = 5^{k-1} + 2$ is true for $k \in \mathbb{Z}^+$.

$$\begin{aligned}
\text{Then }\ u_{k+1} &= 5u_k - 8 \\
&= 5\left(5^{k-1} + 2\right) - 8 \\
&= 5^{k-1+1} + 10 - 8 \\
&= 5^k + 2 \\
&= 5^{k+1-1} + 2
\end{aligned}$$

Therefore, the general statement, $u_n = 5^{n-1} + 2$ is true when $n = k + 1$.

If u_n is true when $n = k$, then it has been shown that $u_n = 5^{n-1} + 2$ is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise C, Question 4

Question:

Given that $u_{n+1} = 3u_n + 1$, $u_1 = 1$, prove by induction that $u_n = \frac{3^n - 1}{2}$.

Solution:

$n = 1$; $u_1 = \frac{3^1 - 1}{2} = \frac{2}{2} = 1$, as given.

$n = 2$; $u_2 = \frac{3^2 - 1}{2} = \frac{8}{2} = 4$, from the general statement.

and $u_2 = 3u_1 + 1 = 3(1) + 1 = 4$, from the recurrence relation.

So u_n is true when $n = 1$ and also true when $n = 2$.

Assume that for $n = k$ that, $u_k = \frac{3^k - 1}{2}$ is true for $k \in \mathbb{Z}^+$.

$$
\begin{aligned}
\text{Then } u_{k+1} &= 3u_k + 1 \\
&= 3\left(\frac{3^k - 1}{2}\right) + 1 \\
&= \left(\frac{3(3^k) - 3}{2}\right) + \frac{2}{2} \\
&= \frac{3^{k+1} - 3 + 2}{2} \\
&= \frac{3^{k+1} - 1}{2}
\end{aligned}
$$

Therefore, the general statement, $u_n = \frac{3^n - 1}{2}$ is true when $n = k + 1$.

If u_n is true when $n = k$, then it has been shown that $u_n = \frac{3^n - 1}{2}$ is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise C, Question 5

Question:

Given that $u_{n+2} = 5u_{n+1} - 6u_n,\ u_1 = 1,\ u_2 = 5$ prove by induction that $u_n = 3^n - 2^n$.

Solution:

$n = 1$; $u_1 = 3^1 - 2^1 = 3 - 2 = 1$, as given.

$n = 2$; $u_2 = 3^2 - 2^2 = 9 - 4 = 5$, as given.

$n = 3$; $u_3 = 3^3 - 2^3 = 27 - 8 = 19$, from the general statement.

and $u_3 = 5u_2 - 6u_1 = 5(5) - 6(1)$

$= 25 - 6 = 19$, from the recurrence relation.

So u_n is true when $n = 1$, $n = 2$ and also true when $n = 3$.

Assume that for $n = k$ and $n = k + 1$,

both $u_k = 3^k - 2^k$ and $u_{k+1} = 3^{k+1} - 2^{k+1}$ are true for $k \in \mathbb{Z}^+$.

$$\begin{aligned}
\text{Then }\ u_{k+2} &= 5u_{k+1} - 6u_k \\
&= 5\left(3^{k+1} - 2^{k+1}\right) - 6\left(3^k - 2^k\right) \\
&= 5\left(3^{k+1}\right) - 5\left(2^{k+1}\right) - 6\left(3^k\right) + 6\left(2^k\right) \\
&= 5\left(3^{k+1}\right) - 5\left(2^{k+1}\right) - 2\left(3^1\right)\left(3^k\right) + 3\left(2^1\right)\left(2^k\right) \\
&= 5\left(3^{k+1}\right) - 5\left(2^{k+1}\right) - 2\left(3^{k+1}\right) + 3\left(2^{k+1}\right) \\
&= 3\left(3^{k+1}\right) - 2\left(2^{k+1}\right) \\
&= \left(3^1\right)\left(3^{k+1}\right) - \left(2^1\right)\left(2^{k+1}\right) \\
&= 3^{k+2} - 2^{k+2}
\end{aligned}$$

Therefore, the general statement, $u_n = 3^n - 2^n$ is true when $n = k + 2$.

If u_n is true when $n = k$ and $n = k + 1$ then it has been shown that $u_n = 3^n - 2^n$ is also true when $n = k + 2$. As u_n is true for $n = 1$, $n = 2$ and $n = 3$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise C, Question 6

Question:

Given that $u_{n+2} = 6u_{n+1} - 9u_n,\ u_1 = -1,\ u_2 = 0$, prove by induction that $u_n = (n-2)3^{n-1}$.

Solution:

$n = 1$; $u_1 = (1-2)3^{1-1} = (-1)(1) = -1$, as given.

$n = 2$; $u_2 = (2-2)3^{2-1} = (0)(3) = 0$, as given.

$n = 3$; $u_3 = (3-2)3^{3-1} = (1)(9) = 9$, from the general statement.

$$\begin{aligned}\text{and } u_3 &= 6u_2 - 9u_1 = 6(0) - 9(-1)\\ &= 0 - -9 = 9, \text{ from the recurrence relation.}\end{aligned}$$

So u_n is true when $n = 1, n = 2$ and also true when $n = 3$.

Assume that for $n = k$ and $n = k + 1$,

both $u_k = (k-2)3^{k-1}$

and $u_{k+1} = (k+1-2)3^{k+1-1} = (k-1)3^k$ are true for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\text{Then } u_{k+2} &= 6u_{k+1} - 9u_k\\ &= 6\left((k-1)3^k\right) - 9\left((k-2)3^{k-1}\right)\\ &= 6(k-1)\left(3^k\right) - 3(k-2).3\left(3^{k-1}\right)\\ &= 6(k-1)\left(3^k\right) - 3(k-2)\left(3^{k-1+1}\right)\\ &= 6(k-1)\left(3^k\right) - 3(k-2)\left(3^k\right)\\ &= \left(3^k\right)[6(k-1) - 3(k-2)]\\ &= \left(3^k\right)[6k - 6 - 3k + 6]\\ &= 3k\left(3^k\right)\\ &= k\left(3^{k+1}\right)\\ &= (k+2-2)\left(3^{k+2-1}\right)\end{aligned}$$

Therefore, the general statement, $u_n = (n-2)3^{n-1}$ is true when $n = k + 2$.

If u_n is true when $n = k$ and $n = k + 1$ then it has been shown that $u_n = (n-2)3^{n-1}$ is also true when $n = k + 2$. As u_n is true for $n = 1, n = 2$ and $n = 3$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise C, Question 7

Question:

Given that $u_{n+2} = 7u_{n+1} - 10u_n$, $u_1 = 1$, $u_2 = 8$, prove by induction that $u_n = 2(5^{n-1}) - 2^{n-1}$.

Solution:

$n = 1$; $u_1 = 2(5^0) - (2^0) = 2 - 1 = 1$, as given.

$n = 2$; $u_2 = 2(5^1) - (2^1) = 10 - 2 = 8$, as given.

$n = 3$; $u_3 = 2(5^2) - (2^2) = 50 - 4 = 46$, from the general statement.

and $u_3 = 7u_2 - 10u_1 = 7(8) - 10(1)$
$= 56 - 10 = 46$, from the recurrence relation.

So u_n is true when $n = 1$, $n = 2$ and also true when $n = 3$.

Assume that for $n = k$ and $n = k + 1$,

both $u_k = 2(5^{k-1}) - 2^{k-1}$

and $u_{k+1} = 2(5^{k+1-1}) - 2^{k+1-1} = 2(5^k) - 2^k$ are true for $k \in \mathbb{Z}^+$.

$$\begin{aligned}
\text{Then } u_{k+2} &= 7u_{k+1} - 10u_k \\
&= 7\left(2(5^k) - 2^k\right) - 10\left(2(5^{k-1}) - 2^{k-1}\right) \\
&= 14\left(5^k\right) - 7\left(2^k\right) - 20\left(5^{k-1}\right) + 10\left(2^{k-1}\right) \\
&= 14\left(5^k\right) - 7\left(2^k\right) - 4\left(5^1\right)\left(5^{k-1}\right) + 5\left(2^1\right)\left(2^{k-1}\right) \\
&= 14\left(5^k\right) - 7\left(2^k\right) - 4\left(5^{k-1+1}\right) + 5\left(2^{k-1+1}\right) \\
&= 14\left(5^k\right) - 7\left(2^k\right) - 4\left(5^k\right) + 5\left(2^k\right) \\
&= 2\left(5^1\right)\left(5^k\right) - \left(2^1\right)\left(2^k\right) \\
&= 2\left(5^{k+1}\right) - \left(2^{k+1}\right) \\
&= 2\left(5^{k+2-1}\right) - \left(2^{k+2-1}\right)
\end{aligned}$$

Therefore, the general statement, $u_n = 2(5^{n-1}) - 2^{n-1}$ is true when $n = k + 2$.

If u_n is true when $n = k$ and $n = k + 1$ then it has been shown that $u_n = 2(5^{n-1}) - 2^{n-1}$ is also true when $n = k + 2$. As u_n is true for $n = 1$, $n = 2$ and $n = 3$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise C, Question 8

Question:

Given that $u_{n+2} = 6u_{n+1} - 9u_n$, $u_1 = 3$, $u_2 = 36$, prove by induction that $u_n = (3n-2)3^n$.

Solution:

$n = 1$; $u_1 = (3(1)-2)(3^1) = (1)(3) = 3$, as given.

$n = 2$; $u_2 = (3(2)-2)(3^2) = (4)(9) = 36$, as given.

$n = 3$; $u_3 = (3(3)-2)(3^3) = (7)(27) = 189$, from the general statement.

and $u_3 = 6u_2 - 9u_1 = 6(36) - 9(3)$
$= 216 - 27 = 189$, from the recurrence relation.

So u_n is true when $n = 1, n = 2$ and also true when $n = 3$.

Assume that for $n = k$ and $n = k+1$,

both $u_k = (3k-2)(3^k)$

and $u_{k+1} = (3(k+1)-2)(3^{k+1}) = (3k+1)(3^{k+1})$ are true for $k \in \mathbb{Z}^+$.

$$
\begin{aligned}
\text{Then } u_{k+2} &= 6u_{k+1} - 9u_k \\
&= 6\left((3k+1)(3^{k+1})\right) - 9\left((3k-2)(3^k)\right) \\
&= 6(3k+1)3^1\left(3^k\right) - 9(3k-2)\left(3^k\right) \\
&= 18(3k+1)\left(3^k\right) - 9(3k-2)(3^k) \\
&= 9\left(3^k\right)[2(3k+1) - (3k-2)] \\
&= 9\left(3^k\right)[6k+2-3k+2] \\
&= 9\left(3^k\right)[3k+4] \\
&= 3^2\left(3^k\right)[3k+4] \\
&= (3k+4)\left(3^{k+2}\right) \\
&= (3(k+2)-2)\left(3^{k+2}\right)
\end{aligned}
$$

Therefore, the general statement, $u_n = (3n-2)3^n$ is true when $n = k+2$.

If u_n is true when $n = k$ and $n = k+1$ then it has been shown that $u_n = (3n-2)3^n$ is also true when $n = k+2$. As u_n is true for $n = 1, n = 2$ and $n = 3$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise D, Question 1

Question:

Prove by the method of mathematical induction the following statement for $n \in \mathbb{Z}^+$.

$$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^n = \begin{pmatrix}1 & 2n\\0 & 1\end{pmatrix}$$

Solution:

$$n = 1;\ \text{LHS} = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^1 = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix}1 & 2(1)\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^k = \begin{pmatrix}1 & 2k\\0 & 1\end{pmatrix}$

With $n = k + 1$ the matrix equation becomes

$$\begin{aligned}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^{k+1} &= \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^k \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\\ &= \begin{pmatrix}1 & 2k\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\\ &= \begin{pmatrix}1+0 & 2+2k\\0+0 & 0+1\end{pmatrix}.\\ &= \begin{pmatrix}1 & 2(k+1)\\0 & 1\end{pmatrix}\end{aligned}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise D, Question 2

Question:

Prove by the method of mathematical induction the following statement for $n \in \mathbb{Z}^+$.

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -4n \\ n & -2n+1 \end{pmatrix}$$

Solution:

$$n = 1;\ \text{LHS} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^1 = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 2(1)+1 & -4(1) \\ 1 & -2(1)+1 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^k = \begin{pmatrix} 2k+1 & -4k \\ k & -2k+1 \end{pmatrix}$.

With $n = k + 1$ the matrix equation becomes

$$\begin{aligned}
\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^k \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2k+1 & -4k \\ k & -2k+1 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 6k+3-4k & -8k-4+4k \\ 3k-2k+1 & -4k+2k-1 \end{pmatrix} \\
&= \begin{pmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{pmatrix} \\
&= \begin{pmatrix} 2(k+1)+1 & -4(k+1) \\ (k+1) & -2(k+1)+1 \end{pmatrix}
\end{aligned}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise D, Question 3

Question:

Prove by the method of mathematical induction the following statement for $n \in \mathbb{Z}^+$.

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$$

Solution:

$$n = 1;\ \text{LHS} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^1 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 2^1 & 0 \\ 2^1 - 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$

With $n = k + 1$ the matrix equation becomes

$$\begin{aligned}
\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^k \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2(2^k) + 0 & 0 + 0 \\ 2(2^k) - 2 + 1 & 0 + 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^1(2^k) & 0 \\ 2^1(2^k) - 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix}
\end{aligned}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise D, Question 4

Question:

Prove by the method of mathematical induction the following statement for $n \in \mathbb{Z}^+$.

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{pmatrix}$$

Solution:

$$n = 1;\ \text{LHS} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^1 = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 4(1)+1 & -8(1) \\ 2(1) & 1-4(1) \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k = \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}$.

With $n = k + 1$ the matrix equation becomes

$$\begin{aligned}
\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^{k+1} &= \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \\
&= \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix} \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \\
&= \begin{pmatrix} 20k+5-16k & -32k-8+24k \\ 10k+2-8k & -16k-3+12k \end{pmatrix} \\
&= \begin{pmatrix} 4k+5 & -8k-8 \\ 2k+2 & -4k-3 \end{pmatrix} \\
&= \begin{pmatrix} 4(k+1)+1 & -8(k+1) \\ 2(k+1) & 1-4(k+1) \end{pmatrix}
\end{aligned}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise D, Question 5

Question:

Prove by the method of mathematical induction the following statement for $n \in \mathbb{Z}^+$.

$$\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 5(2^n - 1) \\ 0 & 1 \end{pmatrix}$$

Solution:

$$n = 1;\ \text{LHS} = \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 2^1 & 5(2^1 - 1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 2^k & 5(2^k - 1) \\ 0 & 1 \end{pmatrix}$

With $n = k + 1$ the matrix equation becomes

$$\begin{aligned}
\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^k & 5(2^k - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2(2^k) + 0 & 5(2^k) + 5(2^k - 1) \\ 0 + 0 & 0 + 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^1(2^k) & 5(2^k) + 5(2^k) - 5 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^{k+1} & 5(2^1)(2^k) - 5 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^{k+1} & 5(2^{k+1}) - 5 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^{k+1} & 5(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise E, Question 1

Question:

Prove by induction that $9^n - 1$ is divisible by 8 for $n \in \mathbb{Z}^+$.

Solution:

Let $f(n) = 9^n - 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 9^1 - 1 = 8$, which is divisible by 8.

$\therefore\ f(n)$ is divisible by 8 when $n = 1$.

Assume that for $n = k$,

$f(k) = 9^k - 1$ is divisible by 8 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ f(k+1) &= 9^{k+1} - 1\\ &= 9^k.9^1 - 1\\ &= 9(9^k) - 1\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1) - f(k) &= [9(9^k) - 1] - [9^k - 1]\\ &= 9(9^k) - 1 - 9^k + 1\\ &= 8(9^k)\end{aligned}$$

$$\therefore\ f(k+1) = f(k) + 8(9^k)$$

As both $f(k)$ and $8(9^k)$ are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore $f(n)$ is divisible by 8 when $n = k + 1$.

If $f(n)$ is divisible by 8 when $n = k$, then it has been shown that $f(n)$ is also divisible by 8 when $n = k + 1$. As $f(n)$ is divisible by 8 when $n = 1$, $f(n)$ is also divisible by 8 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise E, Question 2

Question:

The matrix **B** is given by $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

a Find $\mathbf{B}^2$ and $\mathbf{B}^3$.

b Hence write down a general statement for $\boldsymbol{B}^n$, for $n \in \mathbb{Z}^+$.

c Prove, by induction that your answer to part **b** is correct.

Solution:

a

$$\mathbf{B}^2 = \mathbf{BB} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$$
$$\mathbf{B}^3 = \mathbf{B}^2\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+27 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 27 \end{pmatrix}$$

b As $\mathbf{B}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 3^2 \end{pmatrix}$ and $\mathbf{B}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 3^3 \end{pmatrix}$, we suggest that $\mathbf{B}^n = \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix}$.

c

$$n = 1;\ \text{LHS} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$
$$\text{RHS} = \begin{pmatrix} 1 & 0 \\ 0 & 3^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ 0 & 3^k \end{pmatrix}$

With $n = k + 1$ the matrix equation becomes

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 3^k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+3(3^k) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 3^{k+1} \end{pmatrix} \end{aligned}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise E, Question 3

Question:

Prove by induction that for $n \in \mathbb{Z}^+$, that $\sum_{r=1}^{n}(3r+4) = \frac{1}{2}n(3n+11)$.

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1}(3r+4) = 7$$
$$\text{RHS} = \frac{1}{2}(1)(14) = \frac{1}{2}(14) = 7$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k}(3r+4) = \frac{1}{2}k(3k+11)$.

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}
\sum_{r=1}^{k+1}(3r+4) &= 7 + 10 + 13 + \geq + (3k+4) + (3(k+1)+4) \\
&= \frac{1}{2}k(3k+11) + (3(k+1)+4) \\
&= \frac{1}{2}k(3k+11) + (3k+7) \\
&= \frac{1}{2}[k(3k+11) + 2(3k+7)] \\
&= \frac{1}{2}[3k^2 + 11k + 6k + 14] \\
&= \frac{1}{2}[3k^2 + 17k + 14] \\
&= \frac{1}{2}(k+1)(3k+14) \\
&= \frac{1}{2}(k+1)[3(k+1)+11]
\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise E, Question 4

Question:

A sequence $u_1, u_2, u_3, u_4, \geq$ is defined by $u_{n+1} = 5u_n - 3(2^n),\ u_1 = 7$.

a Find the first four terms of the sequence.

b Prove, by induction for $n \in \mathbb{Z}^+$, that $u_n = 5^n + 2^n$.

Solution:

a $u_{n+1} = 5u_n - 3(2^n)$

Given, $u_1 = 7$.

$u_2 = 5u_1 - 3(2^1) = 5(7) - 6 = 35 - 6 = 29$

$u_3 = 5u_2 - 3(2^2) = 5(29) - 3(4) = 145 - 12 = 133$

$u_4 = 5u_3 - 3(2^3) = 5(133) - 3(8) = 665 - 24 = 641$

The first four terms of the sequence are 7, 29, 133, 641.

b

$n = 1$; $u_1 = 5^1 + 2^1 = 5 + 2 = 7$, as given.

$n = 2$; $u_2 = 5^2 + 2^2 = 25 + 4 = 29$, from the general statement.

From the recurrence relation in part (a), $u_2 = 29$.

So u_n is true when $n = 1$ and also true when $n = 2$.

Assume that for $n = k$, $u_k = 5^k + 2^k$ is true for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\text{Then }\ u_{k+1} &= 5u_k - 3(2^k)\\ &= 5(5^k + 2^k) - 3(2^k)\\ &= 5(5^k) + 5(2^k) - 3(2^k)\\ &= 5^1(5^k) + 2^1(2^k)\\ &= 5^{k+1} + 2^{k+1}\end{aligned}$$

Therefore, the general statement, $u_n = 5^n + 2^n$ is true when $n = k + 1$.

If u_n is true when $n = k$, then it has been shown that $u_n = 5^n + 2^n$ is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise E, Question 5

Question:

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$.

a Prove by induction that $\mathbf{A}^n = \begin{pmatrix} 8n+1 & 16n \\ -4n & 1-8n \end{pmatrix}$ for $n \in \mathbb{Z}^+$.

The matrix **B** is given by $\mathbf{B} = (\mathbf{A}^n)^{-1}$

b Hence find **B** in terms of n.

Solution:

a

$$n = 1;\ \text{LHS} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^1 = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 8(1)+1 & 16(1) \\ -4(1) & 1-8(1) \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^k = \begin{pmatrix} 8k+1 & 16k \\ -4k & 1-8k \end{pmatrix}$.

With $n = k+1$ the matrix equation becomes

$$\begin{aligned}
\begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^{k+1} &= \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^k \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix} \\
&= \begin{pmatrix} 8k+1 & 16k \\ -4k & 1-8k \end{pmatrix} \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix} \\
&= \begin{pmatrix} 72k+9-64k & 128k+16-112k \\ -36k-4+32k & -64k-7+56k \end{pmatrix} \\
&= \begin{pmatrix} 8k+9 & 16k+16 \\ -4k-4 & -8k-7 \end{pmatrix} \\
&= \begin{pmatrix} 8(k+1)+1 & 16(k+1) \\ -4(k+1) & 1-8(k+1) \end{pmatrix}
\end{aligned}$$

Therefore the matrix equation is true when $n = k+1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k+1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

b

$$\begin{aligned}
\det(\mathbf{A}^n) &= (8n+1)(1-8n) - -64n^2 \\
&= 8n - 64n^2 + 1 - 8n + 64n^2 \\
&= 1
\end{aligned}$$

$\mathbf{B} = (\mathbf{A}^n)^{-1} = \frac{1}{1}\begin{pmatrix} 1-8n & -16n \\ 4n & 8n+1 \end{pmatrix}$

So $\mathbf{B} = \begin{pmatrix} 1-8n & -16n \\ 4n & 8n+1 \end{pmatrix}$

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Proof by mathematical induction
Exercise E, Question 6

Question:

The function f is defined by $f(n) = 5^{2n-1} + 1$, where $n \in \mathbb{Z}^+$.

a Show that $f(n+1) - f(n) = \mu(5^{2n-1})$, where μ is an integer to be determined.

b Hence prove by induction that f(n) is divisible by 6.

Solution:

a

$$\begin{aligned} f(n+1) &= 5^{2(n+1)-1} + 1 \\ &= 5^{2n+2-1} + 1 \\ &= 5^{2n-1}.5^2 + 1 \\ &= 25(5^{2n-1}) + 1 \end{aligned}$$

$$\begin{aligned} \therefore\ f(n+1) - f(n) &= \left[25(5^{2n-1}) + 1\right] - [5^{2n-1} + 1] \\ &= 25(5^{2n-1}) + 1 - (5^{2n-1}) - 1 \\ &= 24(5^{2n-1}) \end{aligned}$$

Therefore, $\mu = 24$.

b $f(n) = 5^{2n-1} + 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 5^{2(1)-1} + 1 = 5^1 + 1 = 6$, which is divisible by 6.

$\therefore$ f(n) is divisible by 6 when $n = 1$.

Assume that for $n = k$,

$f(k) = 5^{2k-1} + 1$ is divisible by 6 for $k \in \mathbb{Z}^+$.

Using (a), $f(k+1) - f(k) = 24(5^{2k-1})$

$\therefore\ f(k+1) = f(k) + 24(5^{2k-1})$

As both f(k) and $24(5^{2k-1})$ are divisible by 6 then the sum of these two terms must also be divisible by 6. Therefore f(n) is divisible by 6 when $n = k + 1$.

If f(n) is divisible by 6 when $n = k$, then it has been shown that f(n) is also divisible by 6 when $n = k + 1$. As f(n) is divisible by 6 when $n = 1$, f(n) is also divisible by 6 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise E, Question 7

Question:

Use the method of mathematical induction to prove that $7^n + 4^n + 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

Solution:

Let $\text{f}(n) = 7^n + 4^n + 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ \text{f}(1) = 7^1 + 4^1 + 1 = 7 + 4 + 1 = 12$, which is divisible by 6.

$\therefore\ \text{f}(n)$ is divisible by 6 when $n = 1$.

Assume that for $n = k$,

$\text{f}(k) = 7^k + 4^k + 1$ is divisible by 6 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ \text{f}(k+1) &= 7^{k+1} + 4^{k+1} + 1\\ &= 7^k.7^1 + 4^k.4^1 + 1\\ &= 7(7^k) + 4(4^k) + 1\end{aligned}$$

$$\begin{aligned}\therefore\ \text{f}(k+1) - \text{f}(k) &= [7(7^k) + 4(4^k) + 1] - [7^k + 4^k + 1]\\ &= 7(7^k) + 4(4^k) + 1 - 7^k - 4^k - 1\\ &= 6(7^k) + 3(4^k)\\ &= 6(7^k) + 3(4^{k-1}).4^1\\ &= 6(7^k) + 12(4^{k-1})\\ &= 6[7^k + 2(4)^{k-1}]\end{aligned}$$

$\therefore\ \text{f}(k+1) = \text{f}(k) + 6[7^k + 2(4)^{k-1}]$

As both $\text{f}(k)$ and $6[7^k + 2(4)^{k-1}]$ are divisible by 6 then the sum of these two terms must also be divisible by 6.

Therefore $\text{f}(n)$ is divisible by 6 when $n = k + 1$.

If $\text{f}(n)$ is divisible by 6 when $n = k$, then it has been shown that $\text{f}(n)$ is also divisible by 6 when $n = k + 1$. As $\text{f}(n)$ is divisible by 6 when $n = 1$, $\text{f}(n)$ is also divisible by 6 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise E, Question 8

Question:

A sequence $u_1, u_2, u_3, u_4, \geq$ is defined by $u_{n+1} = \frac{3u_n - 1}{4}$, $u_1 = 2$.

a Find the first five terms of the sequence.

b Prove, by induction for $n \in \mathbb{Z}^+$, that $u_n = 4\left(\frac{3}{4}\right)^n - 1$.

Solution:

a $u_{n+1} = \frac{3u_n - 1}{4}$.

Given, $u_1 = 2$

$$u_2 = \frac{3u_1 - 1}{4} = \frac{3(2) - 1}{4} = \frac{5}{4}$$

$$u_3 = \frac{3u_2 - 1}{4} = \frac{3\left(\frac{5}{4}\right) - 1}{4} = \frac{\frac{11}{4}}{4} = \frac{11}{16}$$

$$u_4 = \frac{3u_3 - 1}{4} = \frac{3\left(\frac{11}{16}\right) - 1}{4} = \frac{\frac{17}{16}}{4} = \frac{17}{64}$$

$$u_5 = \frac{3u_4 - 1}{4} = \frac{3\left(\frac{17}{64}\right) - 1}{4} = \frac{-\frac{13}{64}}{4} = -\frac{13}{256}$$

The first five terms of the sequence are $2, \frac{5}{4}, \frac{11}{16}, \frac{17}{64}, -\frac{13}{256}$.

b

$n = 1$; $u_1 = 4\left(\frac{3}{4}\right)^1 - 1 = 3 - 1 = 2$, as given.

$n = 2$; $u_2 = 4\left(\frac{3}{4}\right)^2 - 1 = \frac{9}{4} - 1 = \frac{5}{4}$, from the general statement.

From the recurrence relation in part (a), $u_2 = \frac{5}{4}$.

So u_n is true when $n = 1$ and also true when $n = 2$.

Assume that for $n = k$, $u_k = 4\left(\frac{3}{4}\right)^k - 1$ is true for $k \in \mathbb{Z}^+$.

$$\begin{aligned}
\text{Then } u_{k+1} &= \frac{3u_k - 1}{4} \\
&= \frac{3\left[4\left(\frac{3}{4}\right)^k - 1\right] - 1}{4} \\
&= \frac{3}{4}\left[4\left(\frac{3}{4}\right)^k - 1\right] - \frac{1}{4} \\
&= 4\left(\frac{3}{4}\right)^1\left(\frac{3}{4}\right)^k - \frac{3}{4} - \frac{1}{4} \\
&= 4\left(\frac{3}{4}\right)^{k+1} - 1
\end{aligned}$$

Therefore, the general statement, $u_n = 4\left(\frac{3}{4}\right)^n - 1$ is true when $n = k + 1$.

If u_n is true when $n = k$, then it has been shown that $u_n = 4\left(\frac{3}{4}\right)^n - 1$ is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise E, Question 9

Question:

A sequence $u_1, u_2, u_3, u_4, \geq$ is defined by $u_n = 3^{2n} + 7^{2n-1}$.

a Show that $u_{n+1} - 9u_n = \lambda(7^{2k-1})$, where λ is an integer to be determined.

b Hence prove by induction that u_n is divisible by 8 for all positive integers n.

Solution:

a

$$\begin{aligned} u_{n+1} &= 3^{2(n+1)} + 7^{2(n+1)-1} \\ &= 3^{2n}(3^2) + 7^{2n+2-1} \\ &= 3^{2n}(3^2) + 7^{2n-1}(7^2) \\ &= 9(3^{2n}) + 49(7^{2n-1}) \end{aligned}$$

$$\begin{aligned} \therefore\ u_{n+1} - 9u_n &= [9(3^{2n}) + 49(7^{2n-1})] - 9[3^{2n} + 7^{2n-1}] \\ &= 9(3^{2n}) + 49(7^{2n-1}) - 9(3^{2n}) - 9(7^{2n-1}) \\ &= 40(7^{2n-1}) \end{aligned}$$

Therefore, $\lambda = 40$.

b $u_n = 3^{2n} + 7^{2n-1}$, where $n \in \mathbb{Z}^+$.

$\therefore\ u_1 = 3^{2(1)} - 7^{2(1)-1} = 3^2 + 7^1 = 16$, which is divisible by 8.

$\therefore\ u_n$ is divisible by 8 when $n = 1$.

Assume that for $n = k$,

$u_k = 3^{2k} + 7^{2k-1}$ is divisible by 8 for $k \in \mathbb{Z}^+$.

Using (a), $u_{k+1} - 9u_k = 40(7^{2k-1})$

$\therefore\ u_{k+1} = 9u_k + 40(7^{2k-1})$

As both $9u_k$ and $40(7^{2k-1})$ are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore u_n is divisible by 8 when $n = k + 1$.

If u_n is divisible by 8 when $n = k$, then it has been shown that u_n is also divisible by 8 when $n = k + 1$. As u_n is divisible by 8 when $n = 1$, u_n is also divisible by 8 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise E, Question 10

Question:

Prove by induction, for all positive integers n, that

$(1 \times 5) + (2 \times 6) + (3 \times 7) + \geq + n(n+4) = \frac{1}{6}n(n+1)(2n+13)$.

Solution:

The identity $(1 \times 5) + (2 \times 6) + (3 \times 7) + \geq + n(n+4) = \frac{1}{6}n(n+1)(2n+13)$.

can be rewritten as $\sum_{r=1}^{n} r(r+4) = \frac{1}{6}n(n+1)(2n+13)$.

$$\begin{aligned} n = 1;\ \text{LHS} &= \sum_{r=1}^{1} r(r+4) = 1(5) = 5 \\ \text{RHS} &= \frac{1}{6}(1)(2)(15) = \frac{1}{6}(30) = 5 \end{aligned}$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} r(r+4) = \frac{1}{6}k(k+1)(2k+13)$.

With $n = k+1$ terms the summation formula becomes:

$$\begin{aligned} \sum_{r=1}^{k+1} r(r+4) &= 1(5) + 2(6) + 3(7) + \geq + k(k+4) + (k+1)(k+5) \\ &= \frac{1}{6}k(k+1)(2k+13) + (k+1)(k+5) \\ &= \frac{1}{6}(k+1)[k(2k+13) + 6(k+5)] \\ &= \frac{1}{6}(k+1)[2k^2 + 13k + 6k + 30] \\ &= \frac{1}{6}(k+1)[2k^2 + 19k + 30] \\ &= \frac{1}{6}(k+1)(k+2)(2k+15) \\ &= \frac{1}{6}(k+1)(k+1+1)[2(k+1)+13] \end{aligned}$$

Therefore, summation formula is true when $n = k+1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k+1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Examination style paper
Exercise A, Question 1

Question:

Use the standard results for $\sum_{r=1}^{n} r$ and for $\sum_{r=1}^{n} r^2$ to show that, for all positive integers n, $\sum_{r=1}^{n}(r+1)(3r+2) = n(an^2+bn+c)$, where the values of a, b and c should be stated.

Solution:

$$\sum_{r=1}^{n}(r+1)(3r+2) = \sum_{r=1}^{n}(3r^2+5r+2)$$

Multiply out brackets first

$$= 3\sum_{r=1}^{n} r^2 + 5\sum_{r=1}^{n} r + 2\sum_{r=1}^{n} 1$$

Split into three separate parts to isolate $\sum r^2$, $\sum r$ and $\sum 1$

$$= 3\frac{n}{6}(n+1)(2n+1) + 5\frac{n}{2}(n+1) + 2n$$

Use standard formulae for $\sum r^2$, $\sum r$ and remember that $\sum_{r=1}^{n} 1 = n$.

$$= \frac{n}{2}[(n+1)(2n+1) + 5(n+1) + 4]$$

Take out factor $\frac{n}{2}$

$$= \frac{n}{2}\left[2n^2+3n+1+5n+5+4\right]$$

Multiply out the terms in the bracket.

$$= \frac{n}{2}\left[2n^2+8n+10\right]$$

Simplify the bracket.

$$= n\left[n^2+4n+5\right]$$

Take out factor of 2 from bracket which will then be ‘cancelled’ by the $\frac{1}{2}$ term to give the answer.

So $a = 1$, $b = 4$ and $c = 5$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Examination style paper
Exercise A, Question 2

Question:

$f(x) = x^3 + 3x - 6$

The equation $f(x) = 0$ has a root α in the interval [1, 1.5].

a Taking 1.25 as a first approximation to α, apply the Newton–Raphson procedure once to $f(x)$ to obtain a second approximation to α. Give your answer to three significant figures.

b Show that the answer which you obtained is an accurate estimate to three significant figures.

Solution:

a

$$\begin{aligned} f(x) &= x^3 + 3x - 6 \\ f'(x) &= 3x^2 + 3 \end{aligned}$$

Differentiate $f(x)$ to give $f'(x)$

Using the Newton-Raphson procedure with $x_1 = 1.25$

$$x_2 = 1.25 - \frac{f(1.25)}{f'(1.25)}$$

State the Newton-Raphson procedure.

$$\begin{aligned} &= 1.25 - \frac{[1.25^3 + 3 \times 1.25 - 6]}{[3 \times 1.25^2 + 3]} \\ &= 1.25 - \frac{[-0.296875]}{7.6875} \\ &= 1.25 + .0386\ldots \end{aligned}$$

Substitute 1.25.

$$= 1.29 (\text{to } 3 \text{ sf})$$

Give your answer to the required accuracy.

b

$$\begin{aligned} f(1.285) &= -0.023\ldots < 0 \\ f(1.295) &= 0.0567\ldots > 0 \end{aligned}$$

Check the sign of $f(x)$ for the lower and upper bounds of values which round to 1.29 (to 3 sf).

As there is a change of sign and $f(x)$ is continuous the root α satisfies

State 'sign change' and draw a conclusion.

$$1.285 < \alpha < 1.295$$

$\therefore\ \alpha = 1.29$(correct to 3 sf).

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Examination style paper
Exercise A, Question 3

Question:

$\mathbf{R} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$

a Describe fully the geometric transformation represented by each of **R** and **S**.

b Calculate **RS**.

The unit square, U, is transformed by the transformation represented by **S** followed by the transformation represented by **R**.

c Find the area of the image of U after both transformations have taken place.

Solution:

a

R represents a rotation of 135° anti-clockwise about 0.

R takes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$ so is rotation.

S represents an enlargement scale factor $\sqrt{2}$ centre 0

S is of the form $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ so is enlargement with scale factor k.

b

$$\mathbf{RS} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

Use the process of matrix multiplication eg $(ab)\begin{pmatrix} c \\ d \end{pmatrix} = ac + bd$.

c

Determinant of **RS** = 2

$\therefore$ Area scale factor of U is 2.

$\therefore$ Image of U has area 2.

Recall that the determinant of matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$ and that this represents an area scale factor.

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Examination style paper
Exercise A, Question 4

Question:

$f(z) = z^4 + 3z^2 - 6z + 10$

Given that $1 + i$ is a complex root of $f(z) = 0$,

a state a second complex root of this equation.

b Use these two roots to find a quadratic factor of $f(z)$, with real coefficients.

Another quadratic factor of $f(z)$ is $z^2 + 2z + 5$.

c Find the remaining two roots of $f(z) = 0$.

Solution:

a

$1 - i$ is a second root.

This is the conjugate of $1 + i$, and complex roots of polynomial equations with real coefficients occur in conjugate pairs.

b

$[z - (1 + i)][z - (1 - i)]$ is a quadratic factor.

Multiply the two linear factors to give a quadratic factor.

$\therefore\ z^2 - 2z + 2$ is the factor.

c

If $z^2 + 2z + 5 = 0$

$$\begin{aligned} z &= \frac{-2 \pm \sqrt{4 - 20}}{2} \\ &= -1 \pm \frac{1}{2}\sqrt{16}\,i \\ &= -1 \pm 2i \end{aligned}$$

Use the quadratic formula $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Remaining roots are $-1 + 2i$ and $-1 - 2i$.

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Examination style paper
Exercise A, Question 5

Question:

The rectangular hyperbola H has equation $xy = c^2$. The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ lie on the hyperbola H.

a Show that the gradient of the chord PQ is $-\frac{1}{pq}$.

The point $R, \left(3c, \frac{c}{3}\right)$ also lies on H and PR is perpendicular to QR.

b Show that this implies that the gradient of the chord PQ *is* 9.

Solution:

a

The gradient of the chord PQ is $\dfrac{\frac{c}{p}-\frac{c}{q}}{cp-cq}$

Use gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$.

$= c\dfrac{(q-p)}{pq} \div c(p-q)$

Use a common denominator to combine the fractions.

$= c\dfrac{(q-p)}{pq} \times \dfrac{1}{c(p-q)}$

$= -\dfrac{(p-q)}{pq(p-q)}$

Express $(q-p)$ as $-(p-q)$

$= \dfrac{-1}{pq}$

Divide numerator and denominator by the factor $(p-q)$.

b

PR has gradient $\dfrac{-1}{3p}$

QR has gradient $\dfrac{-1}{3q}$

Use the result established in part (a) to deduce these gradients.

These lines are perpendicular

$\therefore \dfrac{-1}{3p} \times \dfrac{-1}{3q} = -1$

Use the condition for perpendicular lines $mm' = -1$.

$\therefore \dfrac{1}{9pq} = -1$

$\therefore \dfrac{1}{pq} = -9$

$\therefore$ Gradient of $PQ = \dfrac{-1}{pq} = 9$.

Find the value of $\dfrac{-1}{pq}$.

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Examination style paper
Exercise A, Question 6

Question:

$\mathbf{M} = \begin{pmatrix} x & 2x-7 \\ -1 & x+4 \end{pmatrix}$

a Find the inverse of matrix **M**, in terms of x, given that **M** is non-singular.

b Show that **M** is a singular matrix for two values of x and state these values.

Solution:

a The determinant of **M** is

$$\begin{aligned} & x(x+4)-(-1)(2x-7) \\ & = x^2+4x+2x-7 \\ & = x^2+6x-7 \end{aligned}$$

The inverse of **M** is

$$\frac{1}{x^2+6x-7}\begin{pmatrix} x+4 & 7-2x \\ 1 & x \end{pmatrix}$$

Use the result that the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

b **M** is singular when

$$\begin{aligned} & x^2+6x-7=0 \\ & \text{ie: } (x+7)(x-1)=0 \\ & \therefore\ x=-7 \text{ or } 1. \end{aligned}$$

Put the value of the determinant of **M** equal to zero.

Then solve the quadratic equation.

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Examination style paper
Exercise A, Question 7

Question:

The complex numbers z and w are given by $z = \frac{7 - \mathrm{i}}{1 - \mathrm{i}}$, and $w = \mathrm{i}z$.

a Express z and w in the form $a + \mathrm{i}b$, where a and b are real numbers.

b Find the argument of w in radians to two decimal places.

c Show z and w on an Argand diagram

d Find $|z - w|$.

Solution:

a

$$z = \frac{7-\mathrm{i}}{1-\mathrm{i}} = \frac{(7-\mathrm{i})(1+\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})} = \frac{8+6\mathrm{i}}{2} = 4 + 3\mathrm{i}$$

Multiply numerator and denominator by the conjugate of $1 - \mathrm{i}$.

Remember $\mathrm{i}^2 = -1$

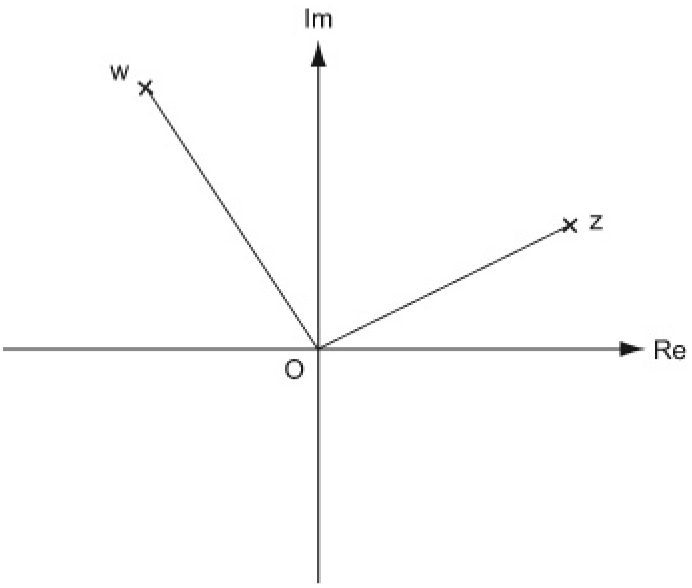
$$w = 1z = \mathrm{i}(4+3\mathrm{i}) = -3 + 4\mathrm{i}$$

b

$$\arg\ w = \pi - \left(\tan^{-1} 4/3\right) = 2.21$$

As w is in the second quadrant in the Argand diagram.

c



d

$$\begin{aligned} z-w &= 7-\mathrm{i} \\ |z-w| &= \sqrt{7^2+(-1)^2} \\ &= \sqrt{50} \\ &= 5\sqrt{2}. \end{aligned}$$

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Examination style paper
Exercise A, Question 8

Question:

The parabola C has equation $y^2 = 16x$.

a Find the equation of the normal to C at the point P, (1, 4).

The normal at P meets the directrix to the parabola at the point Q.

b Find the coordinates of Q.

c Give the coordinates of the point R on the parabola, which is equidistant from Q and from the focus of C.

Solution:

a

$$y^2 = 16x \Rightarrow y = 4x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = 4 \times \frac{1}{2}x^{\frac{-1}{2}}$$
$$= 2x^{\frac{-1}{2}}$$

At (1, 4) gradient is 2

$\therefore$ Gradient of normal is $\frac{-1}{2}$

The equation of the normal is $y - 4 = \frac{-1}{2}(x - 1)$

ie: $y = \frac{-1}{2}x + 4\frac{1}{2}$

Find the gradient of the curve at (1, 4).

Use $mm' = -1$ as the normal is perpendicular to the curve.

Use $y - y_1 = m(x - x_1)$

b

The directrix has equation $x = -4$.

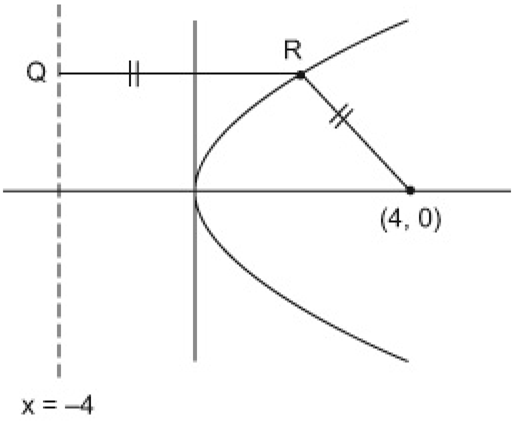
Substitute $x = -4$ into normal equation

$\therefore$ $y = 6\frac{1}{2}$

So Q is the point $\left(-4, 6\frac{1}{2}\right)$.

The directrix of the parabola $y^2 = 4ax$ has equation $x = -a$.

c



$$\text{At } R \; y = 6\frac{1}{2}$$

$$\therefore \left(6\frac{1}{2}\right)^2 = 16x$$

$$\therefore x = \frac{6\frac{1}{2} \times 6\frac{1}{2}}{16} = \frac{169}{64}$$

So R is the point $\left(\frac{169}{64}, \frac{13}{2}\right)$

The point R must have the same y co-ordinate as the point Q.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Examination style paper
Exercise A, Question 9

Question:

a Use the method of mathematical induction to prove that, for $n \varepsilon \mathbb{Z}^+$,

$$\sum_{r=1}^{n} r + \left(\frac{1}{2}\right)^{r-1} = \frac{1}{2}(n^2 + n + 4) - \left(\frac{1}{2}\right)^{n-1}.$$

b $f(n) = 3^{n+2} + (-1)^n 2^n$, $n \varepsilon \mathbb{Z}^+$.

By considering $2f(n+1) - f(n)$ and using the method of mathematical induction prove that, for $n \varepsilon \mathbb{Z}^+$, $3^{n+2} + (-1)^n 2^n$ is divisible by 5.

Solution:

a Let $n = 1$

$$LHS \quad = 1 + \left(\frac{1}{2}\right)^0 = 1 + 1 = 2$$

$$\begin{aligned} RHS \quad &= \frac{1}{2}\left(1^2 + 1 + 4\right) - \left(\frac{1}{2}\right)^0 \\ &= \frac{1}{2} \times 6 - 1 = 2 \end{aligned}$$

Show that the result is true when $n = 1$.

$\therefore$ $LHS = RHS$ so result is true for $n = 1$

Assume that the result is true for $n = k$

ie: $\displaystyle\sum_{r=1}^{k}\left[r + \left(\frac{1}{2}\right)^{r-1}\right] = \frac{1}{2}\left(k^2 + k + 4\right) - \left(\frac{1}{2}\right)^{k-1}$

Add $(k+1) + \left(\frac{1}{2}\right)^k$ to each side.

Show that assuming the result is true for $n = k$ implies that it is also true for $n = k + 1$

$$\begin{aligned} \therefore \sum_{r=1}^{k+1} r + \left(\frac{1}{2}\right)^{r-1} \quad &= \frac{1}{2}\left(k^2 + k + 4\right) + (k+1) - \left(\frac{1}{2}\right)^{k-1} + \left(\frac{1}{2}\right)^k \\ &= \frac{1}{2}\left(k^2 + k + 4 + 2k + 2\right) + \left(\frac{1}{2}\right)^{k-1}\left(-1 + \frac{1}{2}\right) \\ &= \frac{1}{2}\left(k^2 + 3k + 6\right) - \frac{1}{2}\left(\frac{1}{2}\right)^{k-1} \\ &= \frac{1}{2}\left((k+1)^2 + (k+1) + 4\right) - \left(\frac{1}{2}\right)^k \end{aligned}$$

Collect the similar terms together.

ie : $\displaystyle\sum_{r=1}^{n} r + \left(\frac{1}{2}\right)^{r-1} \quad = \frac{1}{2}\left(n^2 + n + 4\right) - \left(\frac{1}{2}\right)^{n-1}$

where $n = k + 1$

ie: Result is implied for $n = k + 1$.

$\therefore$ By induction, as result is true for $n = 1$ then it is implied for $n = 2$, $n = 3$, etc… ie: for all positive integer values for n.

Conclude that this implies by induction that the result is true for all positive integers.

b

$f(n) = 3^{n+2} + (-1)^n 2^n \; n \in Z^+$

Let $n = 1$

$$\begin{aligned} f(1) &= 3^3 + (-1)^1 2^1 \\ &= 27 - 2 \\ &= 25 \end{aligned}$$

Show that the result is true when $n = 1$.

This is divisible by 5.

Let $f(k)$ be divisible by 5

Assume that $f(k)$ is divisible by 5

ie: $3^{k+2} + (-1)^k 2^k = 5A$ *

Consider

$$2f(k+1) - f(k) = 2.3^{k+3} + 2(-1)^{k+1}2^{k+1} - 3^{k+2} - (-1)^k 2^k$$

Follow the hint given in the question

$$\begin{aligned} &= 3^{k+2}[2.3 - 1] + 2^k(-1)^k[-4 - 1] \\ &= 3^{k+2} \times 5 - 5.(-1)^k 2^k \\ &= 5\left(3^{k+2} - (-1)^k 2^k\right). \end{aligned}$$

Collect similar terms together and look for common factor of 5.

$\therefore$ $2f(k+1) - f(k)$ is divisible by 5.

$= 5B$

$$\begin{aligned} \therefore\ 2f(k+1) &= 5B + f(k) \\ &= 5(B + a) \end{aligned}$$

As $f(k)$ and $2f(k+1) - f(k)$ are each divisible by 5, deduce that $f(k+1)$ is also divisible by 5.

ie: $2f(k+1)$ is divisible by $5 \Rightarrow f(k+1)$ is divisible by 5.

So by induction as f(1) is divisible by 5 then so is f(2) and so is f(3) …. and by induction $f(n)$ is divisible by 5 for all positive integers n.

Use induction to complete your proof.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 1

Question:

$z_1 = 2 + \mathrm{i}$, $z_2 = 3 + 4\mathrm{i}$. Find the modulus and the tangent of the argument of each of

a $z_1 z_2^*$

b $\frac{z_1}{z_2}$

Solution:

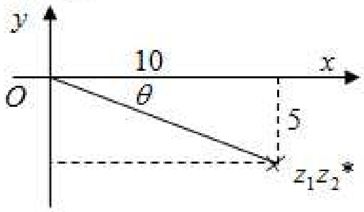
a $z_2^* = 3-4\mathrm{i}$

z^* is the symbol for the conjugate complex number of z.
If $z = a+\mathrm{i}b$, then $z^* = a-\mathrm{i}b$.

$$z_1 z_2^* = (2+\mathrm{i})(3-4\mathrm{i})$$
$$= 6-8\mathrm{i}+3\mathrm{i}-4\mathrm{i}^2$$
$$= 10-5\mathrm{i}$$

$-4\mathrm{i}^2 = -4\times -1 = +4$

$$|z_1 z_2^*|^2 = 10^2 + (-5)^2 = 125$$
$$|z_1 z_2^*| = \sqrt{125} = 5\sqrt{5}$$



$$\tan\theta = \frac{5}{10} = \frac{1}{2}$$

$z_1 z_2^*$ is in the fourth quadrant.

$$\tan\arg(z_1 z_2^*) = -\frac{1}{2}$$

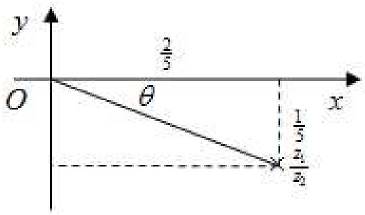
Arguments in the fourth quadrant are negative. The tangents of arguments are negative in the second and fourth quadrants.

b $$\frac{z_1}{z_2} = \frac{2+\mathrm{i}}{3+4\mathrm{i}} \times \frac{3-4\mathrm{i}}{3-4\mathrm{i}}$$
$$= \frac{6-8\mathrm{i}+3\mathrm{i}+4}{25} = \frac{10-5\mathrm{i}}{25}$$
$$= \frac{2}{5} - \frac{1}{5}\mathrm{i}$$

To simplify a quotient you multiply the numerator and denominator by the conjugate complex of the denominator. The conjugate complex of this denominator $3+4\mathrm{i}$ is $3-4\mathrm{i}$.

$$\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{2}{5}\right)^2 + \left(-\frac{1}{5}\right)^2 = \frac{4}{25} + \frac{1}{25} = \frac{5}{25} = \frac{1}{5}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$



$$\tan\theta = \frac{\frac{1}{5}}{\frac{2}{5}} = \frac{1}{2}$$

$\frac{z_1}{z_2}$ is in the fourth quadrant.

$$\tan\arg\left(\frac{z_1}{z_2}\right) = -\frac{1}{2}$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 2

Question:

a Show that the complex number $\frac{2+3\text{i}}{5+\text{i}}$ can be expressed in the form $\lambda(1+\text{i})$, stating the value of λ.

b Hence show that $\left(\frac{2+3\text{i}}{5+\text{i}}\right)^4$ is real and determine its value.

Solution:

a $\frac{2+3\text{i}}{5+\text{i}} \times \frac{5-\text{i}}{5-\text{i}} = \frac{10-2\text{i}+15\text{i}+3}{26}$

$(5+\text{i})(5-\text{i}) = 5^2+1^2 = 26$
You should practise doing such calculations mentally.

$= \frac{13+13\text{i}}{26} = \frac{1}{2}+\frac{1}{2}\text{i}$

$= \frac{1}{2}(1+\text{i})$

$\lambda = \frac{1}{2}$

You use the result from part (a) to simplify the working in part (b).

b $\left(\frac{2+3\text{i}}{5+\text{i}}\right)^4 = \left[\frac{1}{2}(1+\text{i})\right]^4$

$(1+\text{i})^4$ is expanded using the binomial expansion
$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

$= \frac{1}{16}(1+4\text{i}+6\text{i}^2+4\text{i}^3+\text{i}^4)$

$\text{i}^3 = \text{i}^2\times\text{i} = -1\times\text{i} = -\text{i}$
$\text{i}^4 = \text{i}^2\times\text{i}^2 = -1\times-1 = 1$

$= \frac{1}{16}(1+4\text{i}-6-4\text{i}+1)$

$= \frac{1}{16}\times -4 = -\frac{1}{4}$, a real number

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Review Exercise
Exercise A, Question 3

Question:

$z_1 = 5 + \mathrm{i},\ z_2 = -2 + 3\mathrm{i}$

a Show that $|z_1|^2 = 2\,|z_2|^2$.

b Find $\arg(z_1 z_2)$.

Solution:

a $|z_1|^2 = 5^2 + 1^2 = 26$

$|z_2|^2 = (-2)^2 + 3^2 = 4 + 9 = 13$

If $z = a + \mathrm{i}b$, then $|z|^2 = a^2 + b^2$

$26 = 2 \times 13$

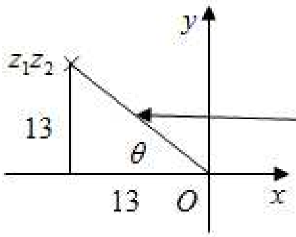
Hence

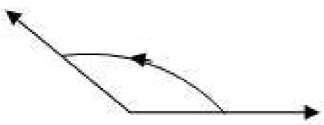
$|z_1|^2 = 2|z_2|^2$, as required.

When you are asked to show or prove a result, you should conclude by saying that you have proved or shown the result. You can write the traditional q.e.d. if you like!

b $z_1 z_2 = (5 + \mathrm{i})(-2 + 3\mathrm{i})$

$= -10 + 15\mathrm{i} - 2\mathrm{i} - 3 = -13 + 13\mathrm{i}$





The argument is the angle with the positive x-axis. Anti-clockwise is positive.

$\tan\theta = \frac{13}{13} = 1 \Rightarrow \theta = \frac{\pi}{4}$

$z_1 z_2$ is in the second quadrant.

$\arg(z_1 z_2) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

As the question has not specified that you should work in radians or degrees. You could work in either and 135° would also be an acceptable answer.

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Review Exercise
Exercise A, Question 4

Question:

a Find, in the form $p+\mathrm{i}q$ where p and q are real, the complex number z which satisfies the equation $\frac{3z-1}{2-\mathrm{i}}=\frac{4}{1+2\mathrm{i}}$.

b Show on a single Argand diagram the points which represent z and z^*.

c Express z and z^* in modulus–argument form, giving the arguments to the nearest degree.

Solution:

a

$$\frac{3z-1}{2-\mathrm{i}}=\frac{4}{1+2\mathrm{i}}$$

$$3z-1=\frac{8-4\mathrm{i}}{1+2\mathrm{i}}\times\frac{1-2\mathrm{i}}{1-2\mathrm{i}}$$

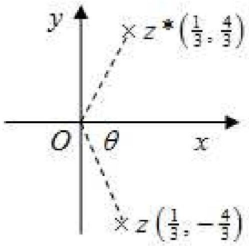
$$=\frac{8-16\mathrm{i}-4\mathrm{i}-8}{5}=\frac{-20\mathrm{i}}{5}=-4\mathrm{i}$$

$$3z=1-4\mathrm{i}$$

$$z=\frac{1}{3}-\frac{4}{3}\mathrm{i}$$

You multiply both sides of the equation by $2-\mathrm{i}$.
Then multiply the numerator and denominator by the conjugate complex of the denominator.

b



You place the points in the Argand diagram which represent conjugate complex numbers symmetrically about the real x-axis.
Label the points so it is clear which is the original number (z) and which is the conjugate (z^*).

c

$$|z|^2=\left(\frac{1}{3}\right)^2+\left(-\frac{4}{3}\right)^3=\frac{1}{9}+\frac{16}{9}=\frac{17}{9}$$

$$|z|=\frac{\sqrt{17}}{3}$$

$$\tan\theta=\frac{\frac{4}{3}}{\frac{1}{3}}=4 \Rightarrow \theta\approx 76°$$

z is in the fourth quadrant.
$\arg z=-76°$, to the nearest degree.

The diagram you have drawn in part (b) shows that z is in the fourth quadrant. There is no need to draw it again.

$$z=\frac{\sqrt{17}}{3}\cos(-76°)+\mathrm{i}\frac{\sqrt{17}}{3}\sin(-76°)$$

$$z^*=\frac{\sqrt{17}}{3}\cos 76°+\mathrm{i}\frac{\sqrt{17}}{3}\sin 76°$$

It is always true that $|z^*|=|z|$ and $\arg z^*=-\arg z$, so you just write down the final answer without further working.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 5

Question:

$z_1 = -1 + \mathrm{i}\sqrt{3}$, $z_2 = \sqrt{3} + \mathrm{i}$

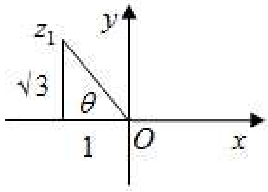
a Find **i** $\arg z_1$ **ii** $\arg z_2$.

b Express $\frac{z_1}{z_2}$ in the form $a + \mathrm{i}b$, where a and b are real, and hence find $\arg\left(\frac{z_1}{z_2}\right)$.

c Verify that $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

Solution:

a i

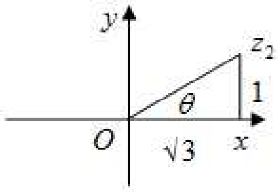


$\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

z_1 is in the second quadrant

$\arg z_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

ii



$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

z_2 is in the second quadrant

$\arg z_2 = \frac{\pi}{6}$

b $\frac{z_1}{z_2} = \frac{-1+i\sqrt{3}}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i}$

$= \frac{-\sqrt{3}+i+3i+\sqrt{3}}{4} = 0+i$

$(\sqrt{3}+i)(\sqrt{3}-i) = (\sqrt{3})^2 - i^2$
$= 3+1 = 4$

Although not strictly in the form $a+ib$, the answer i is acceptable.

$\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$

Any number on the positive imaginary axis has argument $\frac{\pi}{2}$

c $\arg z_1 - \arg z_2 = \frac{2\pi}{3} - \frac{\pi}{6}$, from part (a)

$= \frac{4\pi-\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2} = \arg\left(\frac{z_1}{z_2}\right)$

Hence the relation is satisfied by z_1 and z_2.

Verify means show that the equation is satisfied by the particular numbers in this question.
In part (a), you worked out $\arg z_1$ and $\arg z_2$.
In part (b), you worked out $\arg\left(\frac{z_1}{z_2}\right)$.
You substitute your answers into the equation in part (c) and check that it is correct.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 6

Question:

a Find the two square roots of $3 - 4\text{i}$ in the form $a + \text{i}b$, where a and b are real.

b Show the points representing the two square roots of $3 - 4\text{i}$ in a single Argand diagram.

Solution:

a

$$z^2 = 3 - 4\text{i}$$

The square root of, say, 2 is a root of the equation $z^2 = 2$. The square root of any number k, real or complex, is a root of $z^2 = k$.

Let $z = a + \text{i}b$ where a and b are real.

$$(a + \text{i}b)^2 = 3 - 4\text{i}$$

$$a^2 + 2ab\text{i} - b^2 = 3 - 4\text{i}$$

Equating real parts

$$a^2 - b^2 = 3 \quad ❶$$

Equating imaginary parts

$$2ab = -4 \quad ❷$$

Equating real and imaginary parts gives a pair of simultaneous equations one of which is quadratic and the other linear. The method of solving these is given in Edexcel AS and A-level Modular Mathematics Core Mathematics 1, Chapter 3.

From ❷

$$b = -\frac{4}{2a} = -\frac{2}{a} \quad ❸$$

Substitute ❸ into ❶

$$a^2 - \left(-\frac{2}{a}\right)^2 = 3$$

$$a^2 - \frac{4}{a^2} = 3$$

$$a^4 - 3a^2 - 4 = 0$$

$$(a^2 - 4)(a^2 + 1) = 0$$

The only possible solutions of $a^2 + 1 = 0$ are complex, $a = \pm\text{i}$, and as a is real you must ignore these and only consider the roots of $a^2 - 4 = 0$

$$a^2 = 4$$

$$a = 2, -2$$

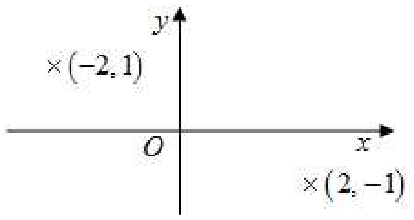
Substitute the values of a into ❸

$$a = 2 \Rightarrow b = -\frac{2}{2} = -1$$

$$a = -2 \Rightarrow b = -\frac{2}{-2} = 1$$

The square roots of $3 - 4\text{i}$ are $2 - \text{i}$ and $-2 + \text{i}$.

b



Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 7

Question:

The complex number z is $-9+17\text{i}$.

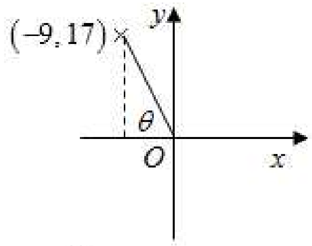
a Show z on an Argand diagram.

b Calculate arg z, giving your answer in radians to two decimal places.

c Find the complex number w for which $zw = 25+35\text{i}$, giving your answer in the form $p+\text{i}q$, where p and q are real.

Solution:

a



b $\tan\theta = \dfrac{17}{9} \Rightarrow \theta = 1.084\ \ldots$

z is in the second quadrant.

$\arg z = \pi - 1.084\ \ldots = 2.057\ \ldots$

$= 2.06$, in radians to 2 d.p.

You have to give your answer to 2 decimal places. To do this accurately you must work to at least 3 decimal places. This avoids rounding errors and errors due to premature approximation.

c $w = \dfrac{25+35\text{i}}{z} = \dfrac{25+35\text{i}}{-9+17\text{i}} = \dfrac{25+35\text{i}}{-9+17\text{i}} \times \dfrac{-9-17\text{i}}{-9-17\text{i}}$

$= \dfrac{-225-425\text{i}-315\text{i}+595}{(-9)^2+17^2}$

$= \dfrac{370-740\text{i}}{370} = 1-2\text{i}$

In this question, the arithmetic gets complicated. Use a calculator to help you with this. However, when you use a calculator, remember to show sufficient working to make your method clear.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 8

Question:

The complex numbers z and w satisfy the simultaneous equations

$2z + \mathrm{i}w = -1,\ \ z - w = 3 + 3\mathrm{i}.$

a Use algebra to find z, giving your answer in the form $a + \mathrm{i}b$, where a and b are real.

b Calculate arg z, giving your answer in radians to two decimal places.

Solution:

a

$2z + \mathrm{i}w = -1$ ❶

$z - w = 3 + 3\mathrm{i}$ ❷

❷ $\times\,\mathrm{i}$ $\quad \mathrm{i}z - \mathrm{i}w = 3\mathrm{i} - 3$ ❸

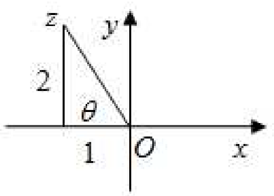
❶ + ❸ $\quad (2+\mathrm{i})z = -4 + 3\mathrm{i}$

$$z = \frac{-4+3\mathrm{i}}{2+\mathrm{i}} \times \frac{2-\mathrm{i}}{2-\mathrm{i}} = \frac{-8+4\mathrm{i}+6\mathrm{i}+3}{5}$$

$$= \frac{-5+10\mathrm{i}}{5} = -1+2\mathrm{i}$$

You use the same method as you learnt for GCSE to solve simultaneous equations. To balance the coefficients of w, you multiply both sides of equation ❷ by i. Adding equations ❶ and ❸ then eliminates w.

b



$\tan\theta = \frac{2}{1} = 2 \Rightarrow \theta = 1.107\ldots$

z is in the second quadrant.

arg $z = \pi - 1.107\ldots = 2.03$, to 2 d.p.

You must work to a least 3 decimal places to obtain an accurate answer to 2 decimal places.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 9

Question:

The complex number z satisfies the equation $\frac{z-2}{z+3\mathrm{i}} = \lambda\mathrm{i},\ \lambda \in \mathbb{R}$.

a Show that $z = \frac{(2-3\lambda)(1+\lambda\mathrm{i})}{1+\lambda^2}$.

b In the case when $\lambda = 1$, find $|z|$ and arg z.

Solution:

a

$$z-2=\lambda\mathrm{i}(z+3\mathrm{i})$$
$$=\lambda\mathrm{i}z-3\lambda$$

$\lambda\mathrm{i}\times 3\mathrm{i}=3\lambda\mathrm{i}^2=-3\lambda$

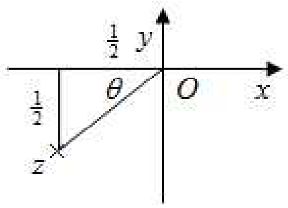
$$z(1-\lambda\mathrm{i})=2-3\lambda$$
$$z=\frac{2-3\lambda}{1-\lambda\mathrm{i}}\times\frac{1+\lambda\mathrm{i}}{1+\lambda\mathrm{i}}$$

You make z the subject of the formula and then multiply the numerator and denominator by $1+\lambda\mathrm{i}$, which is the conjugate complex of $1-\lambda\mathrm{i}$

$$=\frac{(2-3\lambda)(1+\lambda\mathrm{i})}{1+\lambda^2}\text{, as required.}$$

b

$$\lambda=1 \Rightarrow z=\frac{(2-3)(1+\mathrm{i})}{1+1}=-\frac{1}{2}-\frac{1}{2}\mathrm{i}$$
$$|z|^2=\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$
$$|z|=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$



$$\tan\theta=\frac{\frac{1}{2}}{\frac{1}{2}}=1 \Rightarrow \theta=\frac{\pi}{4}$$

z is in the third quadrant.

$$\arg z=-\left(\pi-\frac{\pi}{4}\right)=-\frac{3\pi}{4}$$

The question does not specify radians and $\arg z=-135°$ would be an acceptable answer.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 10

Question:

The complex number z is given by $z = -2 + 2\text{i}$.

a Find the modulus and argument of z.

b Find the modulus and argument of $\frac{1}{z}$.

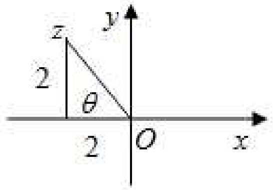
c Show on an Argand diagram the points A, B and C representing the complex numbers z, $\frac{1}{z}$ and $z + \frac{1}{z}$ respectively.

d State the value of $\angle ACB$.

Solution:

a $|z|^2 = (-2)^2 + 2^2 = 4 + 4 = 8$

$|z| = \sqrt{8} = 2\sqrt{2}$



$\tan\theta = \dfrac{2}{2} = 1 \Rightarrow \theta = \dfrac{\pi}{4}$

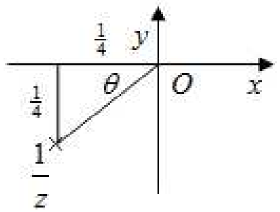
z is in the second quadrant.

$\arg z = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$

b $\dfrac{1}{z} = \dfrac{1}{-2+2\mathrm{i}} \times \dfrac{-2-2\mathrm{i}}{-2-2\mathrm{i}} = \dfrac{-2-2\mathrm{i}}{8} = -\dfrac{1}{4} - \dfrac{1}{4}\mathrm{i}$

$\left|\dfrac{1}{z}\right|^2 = \left(-\dfrac{1}{4}\right)^2 + \left(-\dfrac{1}{4}\right)^2 = \dfrac{1}{16} + \dfrac{1}{16} = \dfrac{1}{8}$

$\left|\dfrac{1}{z}\right| = \dfrac{1}{\sqrt{8}} = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$

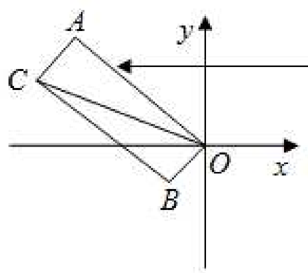


$\tan\theta = \dfrac{\frac{1}{4}}{\frac{1}{4}} = 1 \Rightarrow \theta = \dfrac{\pi}{4}$

z is in the third quadrant.

$\arg z = -\left(\pi - \dfrac{\pi}{4}\right) = -\dfrac{3\pi}{4}$

c



The point C, representing $z + \dfrac{1}{z}$, must be a vertex of the parallelogram which has OA and OB as two of its sides.

In this case, as you have already shown that OA and OB make angles of $\dfrac{\pi}{4}(45°)$ with the negative x-axis, the parallelogram is a rectangle.

d $\angle ACB = 90°$

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Review Exercise
Exercise A, Question 11

Question:

The complex numbers z_1 and z_2 are given by $z_1 = \sqrt{3} + \mathrm{i}$ and $z_2 = 1 - \mathrm{i}$.

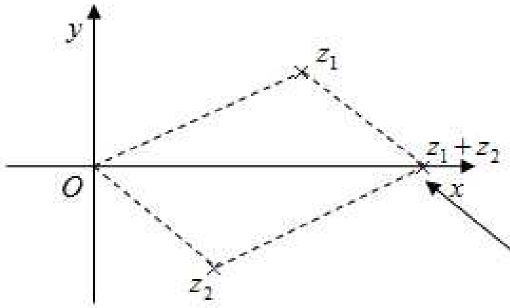
a Show, on an Argand diagram, points representing the complex numbers z_1, z_2 and $z_1 + z_2$.

b Express $\frac{1}{z_1}$ and $\frac{1}{z_2}$, each in the form $a + \mathrm{i}b$, where a and b are real numbers.

c Find the values of the real numbers A and B such that $\frac{A}{z_1} + \frac{B}{z_2} = z_1 + z_2$.

Solution:

a



The point representing z_1+z_2 must form a parallelogram with O and the points representing z_1 and z_2.
$z_1+z_2=\sqrt{3}+1$, which is real, so you must draw the point representing z_1+z_2 on the positive x-axis.

b $\frac{1}{z_1}=\frac{1}{\sqrt{3}+\mathrm{i}}\times\frac{\sqrt{3}-\mathrm{i}}{\sqrt{3}-\mathrm{i}}=\frac{\sqrt{3}-\mathrm{i}}{(\sqrt{3})^2+1^2}$

$$=\frac{\sqrt{3}-\mathrm{i}}{4}=\frac{\sqrt{3}}{4}-\frac{1}{4}\mathrm{i}$$

$$\frac{1}{z_2}=\frac{1}{1-\mathrm{i}}\times\frac{1+\mathrm{i}}{1+\mathrm{i}}=\frac{1+\mathrm{i}}{1^2+1^2}=\frac{1}{2}+\frac{1}{2}\mathrm{i}$$

You use your results in part (b) to simplify the working in part (c). Substitute the answers to part (b) into the printed equation in part (c)

c $\frac{A}{z_1}+\frac{B}{z_2}=z_1+z_2$

$$A\left(\frac{\sqrt{3}}{4}-\frac{1}{4}\mathrm{i}\right)+B\left(\frac{1}{2}+\frac{1}{2}\mathrm{i}\right)=\sqrt{3}+\mathrm{i}+1-\mathrm{i}=\sqrt{3}+1$$

You obtain a pair of simultaneous equations by equating the real and imaginary parts of this equation.

Equating real parts

$$\frac{\sqrt{3}}{4}A+\frac{1}{2}B=\sqrt{3}+1 \quad ❶$$

Equating imaginary parts

$$-\frac{1}{4}A+\frac{1}{2}B=0 \quad ❷$$

❶ – ❷

$$\frac{\sqrt{3}}{4}A+\frac{1}{4}A=\sqrt{3}+1$$

$$\left(\frac{\sqrt{3}+1}{4}\right)A=\sqrt{3}+1$$

$$A=4$$

Substitute in ❷

$$-1+\frac{1}{2}B=0 \Rightarrow B=2$$

$$A=4,\ B=2$$

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Review Exercise
Exercise A, Question 12

Question:

The complex numbers z and w are given by $z=\frac{A}{1-\mathrm{i}}$, $w=\frac{B}{1-3\mathrm{i}}$, where A and B are real numbers. Given that $z+w=\mathrm{i}$,

a find the value of A and the value of B.

b For these values of A and B, find $\tan[\arg(w-z)]$.

Solution:

a

$$z = \frac{A}{1-\mathrm{i}} = \frac{A}{1-\mathrm{i}} \times \frac{1+\mathrm{i}}{1+\mathrm{i}} = \frac{A}{2}(1+\mathrm{i})$$

$$w = \frac{B}{1-3\mathrm{i}} = \frac{B}{1-3\mathrm{i}} \times \frac{1+3\mathrm{i}}{1+3\mathrm{i}} = \frac{B}{10}(1+3\mathrm{i})$$

The expressions for both z and w are fractions with complex denominators. You should remove these, by multiplying both the numerator and denominator by the conjugate complex of the denominator, before substituting into the equation.

$$z + w = \mathrm{i}$$

$$\frac{A}{2}(1+\mathrm{i}) + \frac{B}{10}(1+3\mathrm{i}) = \mathrm{i}$$

When equating the real and complex parts of both sides of the equation, think of the complex number i as $0+1\mathrm{i}$.

Equating real parts

$$\frac{A}{2} + \frac{B}{10} = 0 \quad ❶$$

Equating imaginary parts

$$\frac{A}{2} + \frac{3B}{10} = 1 \quad ❷$$

❷ − ❶

$$\frac{2B}{10} = 1 \Rightarrow B = 5$$

Substitute into ❶

$$\frac{A}{2} + \frac{5}{10} = 0 \Rightarrow \frac{A}{2} = -\frac{1}{2} \Rightarrow A = -1$$

$$A = -1,\ B = 5$$

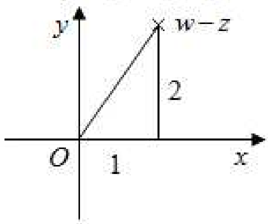
b With these values of A and B

$$z = \frac{-1}{2}(1+\mathrm{i}) = -\frac{1}{2} - \frac{1}{2}\mathrm{i}$$

$$w = \frac{5}{10}(1+3\mathrm{i}) = \frac{1}{2} + \frac{3}{2}\mathrm{i}$$

$$w - z = \frac{1}{2} + \frac{3}{2}\mathrm{i} - \left(-\frac{1}{2} - \frac{1}{2}\mathrm{i}\right)$$

$$= \frac{1}{2} + \frac{3}{2}\mathrm{i} + \frac{1}{2} + \frac{1}{2}\mathrm{i} = 1 + 2\mathrm{i}$$



$$\tan\left[\arg(w-z)\right] = \frac{2}{1} = 2$$

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Review Exercise
Exercise A, Question 13

Question:

a Given that $z = 2 - \mathrm{i}$, show that $z^2 = 3 - 4\mathrm{i}$.

b Hence, or otherwise, find the roots, z_1 and z_2, of the equation $(z + \mathrm{i})^2 = 3 - 4\mathrm{i}$.

c Show points representing z_1 and z_2 on a single Argand diagram.

d Deduce that $|z_1 - z_2| = 2\sqrt{5}$.

e Find the value of $\arg(z_1 + z_2)$.

Solution:

a $z^2 = (2-\mathrm{i})^2 = 4 - 4\mathrm{i} + \mathrm{i}^2$
$= 4 - 4\mathrm{i} - 1$
$= 3 - 4\mathrm{i}$, as required.

You square using the formula $(a-b)^2 = a^2 - 2ab + b^2$

b From part (a), the square roots of $3 - 4\mathrm{i}$ are $2 - \mathrm{i}$ and $-2 + \mathrm{i}$.
Taking square roots of both sides of the equation $(z+\mathrm{i})^2 = 3 - 4\mathrm{i}$

$z + \mathrm{i} = 2 - \mathrm{i} \Rightarrow z = 2 - 2\mathrm{i}$

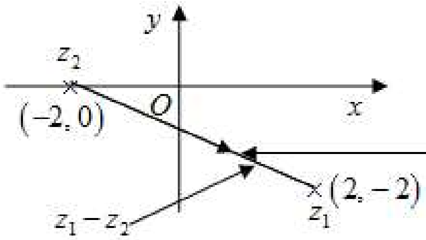
$z + \mathrm{i} = -2 + \mathrm{i} \Rightarrow z = -2$

The square root of any number k, real or complex, is a root of $z^2 = k$. Hence, part (a) shows that one square root of $3 - 4\mathrm{i}$ is $2 - \mathrm{i}$.
If one square root of $3 - 4\mathrm{i}$ is $2 - \mathrm{i}$, then the other is $-(2 - \mathrm{i})$.

$z_1 = 2 - 2\mathrm{i}$, say, and $z_2 = -2$

z_1 and z_2 could be the other way round but that would make no difference to $|z_1 - z_2|$ or $z_1 + z_2$, the expressions you are asked about in parts (d) and (e).

c



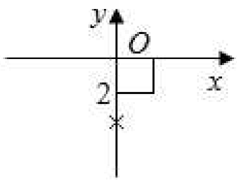
$z_1 - z_2$ can be represented on the diagram you drew in part (c) by the vector joining the point representing z_1 to the point representing z_2. The modulus of $z_1 - z_2$ is then just the length of the line joining these two points and this length can be found using coordinate geometry.

d Using the formula

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= (2-(-2))^2 + (-2-0)^2$$
$$= 4^2 + 2^2 = 20$$

Hence $|z_1 - z_2| = \sqrt{20} = 2\sqrt{5}$

(e) $z_1 + z_2 = 2 - 2\mathrm{i} - 2 = -2\mathrm{i}$



$\arg(z_1 + z_2) = -\dfrac{\pi}{2}$

The argument of any number on the negative imaginary axis is $-\dfrac{\pi}{2}$ or $-90°$.

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Review Exercise
Exercise A, Question 14

Question:

a Find the roots of the equation $z^2 + 4z + 7 = 0$, giving your answers in the form $p + \mathrm{i}\sqrt{q}$, where p and q are integers.

b Show these roots on an Argand diagram.

c Find for each root,

i the modulus,

ii the argument, in radians, giving your answers to three significant figures.

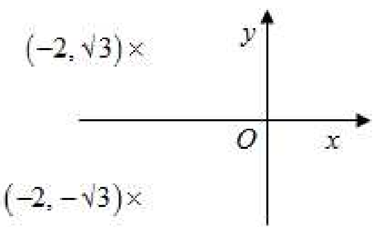
Solution:

a

$$z^2 + 4z = -7$$
$$z^2 + 4z + 4 = -7 + 4 = -3$$
$$(z+2)^2 = -3$$
$$z + 2 = \pm \mathrm{i}\sqrt{3}$$
$$z = -2 + \mathrm{i}\sqrt{3},\ -2 - \mathrm{i}\sqrt{3}$$

You may use any accurate method of solving a quadratic equation. Completing the square works well when the coefficient of z^2 is one and the coefficient of z is even.

b

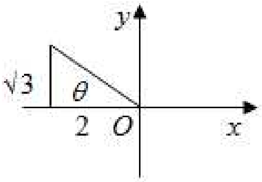


c i

$$|-2 + \mathrm{i}\sqrt{3}|^2 = (-2)^2 + (\sqrt{3})^2 = 4 + 3 = 7$$
$$|-2 + \mathrm{i}\sqrt{3}| = \sqrt{7}$$
$$|-2 - \mathrm{i}\sqrt{3}| = \sqrt{7}$$

The moduli of conjugate complex numbers are the same so you do not have to repeat the working.

c ii



$$\tan\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 0.7137\ldots$$

$-2 + \mathrm{i}\sqrt{3}$ is in the second quadrant

$$\arg(-2 + \mathrm{i}\sqrt{3}) = \pi - 0.7137\ldots$$
$$= 2.43, \text{ to 3 significant figures}$$
$$\arg(-2 - \mathrm{i}\sqrt{3}) = -2.43, \text{ to 3 significant figures}$$

If z and z^* are conjugate complex numbers, then $\arg z^* = -\arg z$. Once you have worked out $\arg z$, you can just write down $\arg z^*$ without further working.

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Review Exercise
Exercise A, Question 15

Question:

Given that $\lambda \varepsilon \mathbb{R}$ and that z and w are complex numbers, solve the simultaneous equations $z - \mathrm{i}w = 2$, $z - \lambda w = 1 - \lambda^2$, giving your answers in the form $a + \mathrm{i}b$, where a, $b \varepsilon \mathbb{R}$, and a and b are functions of λ.

Solution:

$z - \mathrm{i}w = 2 \quad \ldots\ldots\ldots$ ❶

$z - \lambda w = 1 - \lambda^2 \quad \ldots$ ❷

❶ − ❷

$-\mathrm{i}w + \lambda w = 2 - (1 - \lambda^2)$

$w(\lambda - \mathrm{i}) = \lambda^2 + 1$

$$w = \frac{\lambda^2 + 1}{\lambda - \mathrm{i}} \times \frac{\lambda + \mathrm{i}}{\lambda + \mathrm{i}}$$

$$= \frac{\cancel{(\lambda^2 + 1)}(\lambda + \mathrm{i})}{\cancel{(\lambda^2 + 1)}}$$

$$= \lambda + \mathrm{i}$$

Substitute in ❶

$z - \mathrm{i}(\lambda + \mathrm{i}) = 2$

$z = 2 + \mathrm{i}(\lambda + \mathrm{i}) = 2 + \mathrm{i}\lambda - 1 = 1 + \mathrm{i}\lambda$

You solve simultaneous linear equations with complex numbers in exactly the same way as you solved simultaneous equations with real numbers at GCSE. In this case, as the coefficients of z are already balanced, you subtract the equations as they stand to eliminate z.

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Review Exercise
Exercise A, Question 16

Question:

Given that $z_1 = 5 - 2\text{i}$,

a evaluate $|z_1|$, giving your answer as a surd,

b find, in radians to two decimal places, $\arg z_1$.

Given also that z_1 is a root of the equation $z^2 - 10z + c = 0$, where c is a real number,

c find the value of c.

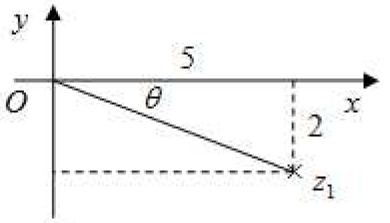
Solution:

a $|z_1|^2 = 5^2 + (-2)^2 = 25 + 4 = 29$

If $z = a + \text{i}b$, then $|z|^2 = a^2 + b^2$

$|z_1| = \sqrt{29}$

b



$\tan\theta = \frac{2}{5} \Rightarrow \theta = 0.3805 \ldots$

is in the fourth quadrant

to 2 decimal places.

c If $z_1 = 5 - 2\text{i}$ is one root of a quadratic equation with real coefficients, then $z_2 = 5 + 2\text{i}$ must be the other root.

If α and β are the roots of a quadratic equation, then the equation must have the form $(z - \alpha)(z - \beta) = 0$.

$$
\begin{aligned}
(z - z_1)(z - z_2) &= (z - 5 + 2\text{i})(z - 5 - 2\text{i}) \\
&= (z - 5)^2 + 4 \\
&= z^2 - 10z + 25 + 4 \\
&= z^2 - 10z + 29 = 0
\end{aligned}
$$

Comparing this with the equation in the question

$c = 29$

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Review Exercise
Exercise A, Question 17

Question:

The complex numbers z and w are given by $z = \frac{5 - 10i}{2 - i}$ and $w = iz$.

a Obtain z and w in the form $p + iq$, where p and q are real numbers.

b Show points representing z and w on a single Argand diagram

The origin O and the points representing z and w are the vertices of a triangle.

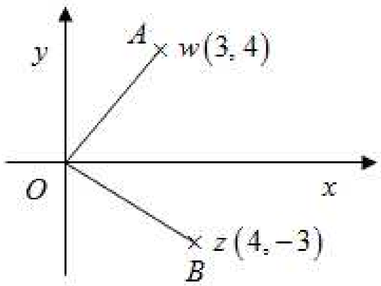
c Show that this triangle is isosceles and state the angle between the equal sides.

Solution:

a

$$z = \frac{5-10i}{2-i} \times \frac{2+i}{2+i}$$
$$= \frac{10+5i-20i+10}{2^2+1^2}$$
$$= \frac{20-15i}{5} = 4-3i$$
$$w = iz = i(4-3i) = 4i - 3i^2 = 3+4i$$

b



c Let A be the point representing w and
B be the point representing z.

$$|w|^2 = 3^2 + 4^2 = 25 \quad \Rightarrow \quad |w| = 5$$
$$|z|^2 = 4^2 + (-3)^2 = 25 \quad \Rightarrow \quad |z| = 5$$

Hence $OA = OB = 5$ and the triangle OAB is isosceles.
The angle between the equal sides, $\angle AOB = 90°$.

As you are only asked to **state** the angle between the equal sides, you do not need to show working. If you cannot see this angle is a right angle or if working was asked for, you could argue:
the gradient of OA, $m = \frac{4}{3}$,
the gradient of OB, $m' = -\frac{3}{4}$.
$mm' = -1$, so the lines are perpendicular.

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Review Exercise
Exercise A, Question 18

Question:

$z_1 = \frac{1+\mathrm{i}}{1-\mathrm{i}},\ z_2 = \frac{\sqrt{2}}{1-\mathrm{i}}$

a Find the modulus and argument of each of the complex numbers z_1 and z_2.

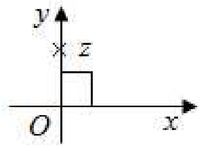
b Plot the points representing z_1, z_2 and $z_1 + z_2$ on a single Argand diagram.

c Deduce from your diagram that $\tan\left(\frac{3\pi}{8}\right) = 1 + \sqrt{2}$.

Solution:

a $z_1 = \frac{1+i}{1-i} \times \frac{1+i}{1+i}$

$= \frac{1+2i+i^2}{1^2+1^2} = \frac{1+2i-1}{2} = \frac{2i}{2} = i$



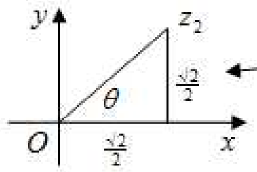
$|z_1| = 1,\ \arg z_1 = \frac{\pi}{2}$

The argument of any number on the positive imaginary axis is $\frac{\pi}{2}$ or $90°$.

$z_2 = \frac{\sqrt{2}}{1-i} \times \frac{1+i}{1+i} = \frac{\sqrt{2}(1+i)}{2} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$|z_2|^2 = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} + \frac{2}{4} = 1$

$|z_2| = 1$



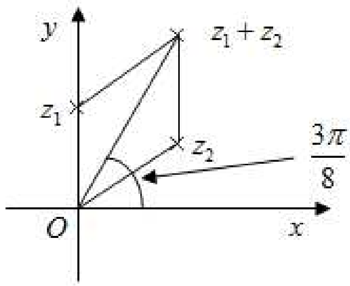
It is worth remembering that any complex number of the form $a + ai$, where $a > 0$, has argument $\frac{\pi}{4}$. This working is then not necessary.

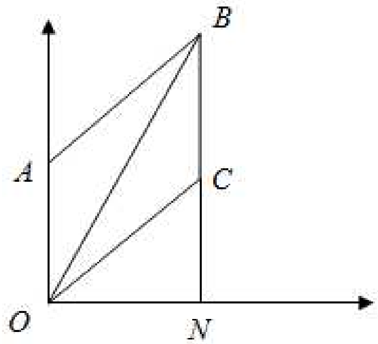
$\tan\theta = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$

z_2 is in the first quadrant

$\arg z_2 = \frac{\pi}{4}$

b





$\angle NOC = 45°$, the argument of z_2

$\angle COA = 90° - 45° = 45°$

$\angle COB = \frac{1}{2}\angle COA = 22\frac{1}{2}°$, the diagonal of a parallelogram bisects the angle

$\angle NOB = 45° + 22\frac{1}{2}° = 67\frac{1}{2}° = \frac{3\pi}{8}$, in radians

$\tan\left(\frac{3\pi}{8}\right) = \tan\left(\arg(z_1+z_2)\right) = \frac{BN}{ON}$

c $z_1 + z_2 = \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2} + 1\right)i$

$\arg(z_1 + z_2) = \frac{3\pi}{8}$

$\tan\left(\frac{3\pi}{8}\right) = \frac{\frac{\sqrt{2}}{2} + 1}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}+2}{\sqrt{2}}$

$= \frac{\sqrt{2}}{\sqrt{2}} + \frac{2}{\sqrt{2}} = 1 + \sqrt{2}$, as required

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Review Exercise
Exercise A, Question 19

Question:

$z_1 = 1 + 2\text{i},\ z_2 = \frac{3}{5} + \frac{4}{5}\text{i}$

a Express in the form $p + q\text{i}$, where $p,\ q \in \mathbb{R}$,

i $z_1 z_2$

ii $\frac{z_1}{z_2}$.

In an Argand diagram, the origin O and the points representing $z_1 z_2$, $\frac{z_1}{z_2}$ and z_3 are the vertices of a rhombus.

b Sketch the rhombus on an Argand diagram.

c Find z_3.

d Show that $|z_3| = \frac{6\sqrt{5}}{5}$.

Solution:

a i $z_1 z_2 = (1+2\mathrm{i})\left(\frac{3}{5}+\frac{4}{5}\mathrm{i}\right)$

$$= \frac{3}{5}+\frac{4}{5}\mathrm{i}+\frac{6}{5}\mathrm{i}-\frac{8}{5} = -1+2\mathrm{i}$$

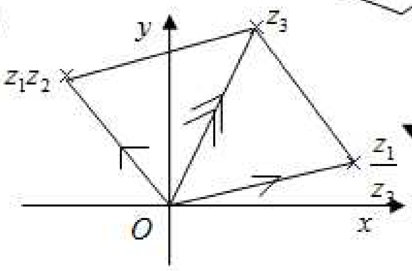
ii $\frac{z_1}{z_2} = \frac{1+2\mathrm{i}}{\frac{3}{5}+\frac{4}{5}\mathrm{i}} \times \frac{\frac{3}{5}-\frac{4}{5}\mathrm{i}}{\frac{3}{5}-\frac{4}{5}\mathrm{i}}$

$$= \frac{\frac{3}{5}-\frac{4}{5}\mathrm{i}+\frac{6}{5}\mathrm{i}+\frac{8}{5}}{1} = \frac{11}{5}+\frac{2}{5}\mathrm{i}$$

$$\left(\frac{3}{5}+\frac{4}{5}\mathrm{i}\right)\left(\frac{3}{5}-\frac{4}{5}\mathrm{i}\right) = \left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2 = \frac{9+16}{25} = 1$$

The relation between $\frac{3}{5}$, $\frac{4}{5}$ and 1 is the well-known 3, 4, 5 relation divided by 5 and, with practice, you can just write down answers like this.

b



On an Argand diagram the sum of two complex numbers can be represented by the diagonal completing the parallelogram, as shown in this diagram. (A rhombus is a special case of a parallelogram.)

c $z_3 = z_1 z_2 + \frac{z_1}{z_2} = -1+2\mathrm{i}+\frac{11}{5}+\frac{2}{5}\mathrm{i}$

$$= \frac{6}{5}+\frac{12}{5}\mathrm{i}$$

d $|z_3|^2 = \left(\frac{6}{5}\right)^2+\left(\frac{12}{5}\right)^2 = \frac{36+144}{25} = \frac{180}{25} = \frac{36\times 5}{25}$

Hence $|z_3| = \frac{6\sqrt{5}}{5}$, as required

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 20

Question:

$z_1 = -30 + 15\text{i}$.

a Find $\arg z_1$, giving your answer in radians to two decimal places.

The complex numbers z_2 and z_3 are given by $z_2 = -3 + p\text{i}$ and $z_3 = q + 3\text{i}$, where p and q are real constants and $p > q$.

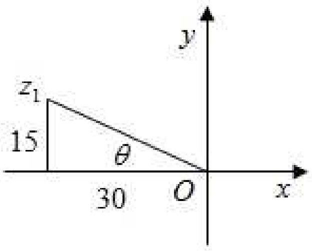
b Given that $z_2 z_3 = z_1$, find the value of p and the value of q.

c Using your values of p and q, plot the points corresponding to z_1, z_2 and z_3 on an Argand diagram.

d Verify that $2z_2 + z_3 - z_1$ is real and find its value.

Solution:

a



$\tan\theta = \frac{15}{30} = \frac{1}{2} \Rightarrow \theta \approx 0.464$

z_1 is in the second quadrant.

$\arg z_1 = \pi - \theta = 2.68$ to 2 d.p.

As you are asked to give your answer to 2 decimal places, you should work to at least 3 decimal places. This avoids rounding errors.

b

$$z_2 z_3 = z_1$$
$$(-3 + p\mathrm{i})(q + 3\mathrm{i}) = -30 + 15\mathrm{i}$$
$$-3q - 9\mathrm{i} + pq\mathrm{i} - 3p = -30 + 15\mathrm{i}$$

Equating real parts

$-3q - 3p = -30 \Rightarrow p + q = 10$ …… ❶

Equating imaginary parts

$-9 + pq = 15 \Rightarrow pq = 24$ … … … ❷

From ❷

$q = \frac{24}{p}$ … … … … ❸

Equating real and imaginary parts gives a pair of simultaneous equations one of which is quadratic and the other linear. The method of solving these is given in Edexcel AS and A-level Modular Mathematics Core Mathematics 1, Chapter 3.

Substitute ❸ into ❶

$$p + \frac{24}{p} = 10$$
$$p^2 - 10p + 24 = (p-4)(p-6) = 0$$
$$p = 4, 6$$

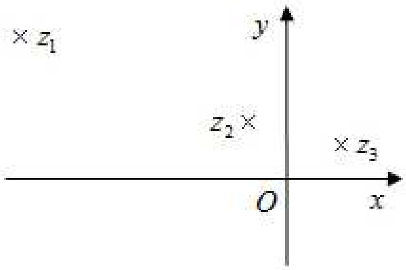
Substituting $p = 4$ into ❶ gives $q = 6$.

As $p > q$ is given, this solution is rejected.

Substituting $p = 6$ into ❶ gives $q = 4$.

$p = 6$, $q = 4$ is the only solution.

c



d $2z_2 + z_3 - z_1 = 2(-3 + 6\mathrm{i}) + 4 + 3\mathrm{i} - (-30 + 15\mathrm{i})$

$= -6 + 12\mathrm{i} + 4 + 3\mathrm{i} + 30 - 15\mathrm{i} = 28$, a real number

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 21

Question:

Given that $z = 1 + \sqrt{3}\mathrm{i}$ and that $\frac{w}{z} = 2 + 2\mathrm{i}$, find

a w in the form $a + \mathrm{i}b$, where $a,\ b \in \mathbb{R}$,

b the argument of w,

c the exact value for the modulus of w.

On an Argand diagram, the point A represents z and the point B represents w.

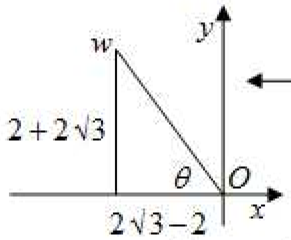
d Draw the Argand diagram, showing the points A and B.

e Find the distance AB, giving your answer as a simplified surd.

Solution:

a $w=(2+2i)z=(2+2i)(1+\sqrt{3}i)$
$=2+2\sqrt{3}i+2i-2\sqrt{3}$
$=(2-2\sqrt{3})+(2+2\sqrt{3})i$

b



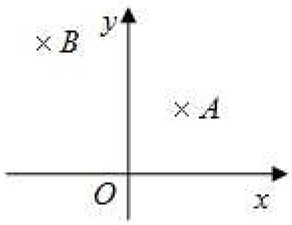
$2-2\sqrt{3}\approx-1.46$ so w is in the second quadrant.
The length of the side is $2\sqrt{3}-2$, not $2-2\sqrt{3}$ as lengths have to be positive.

$\tan\theta=\dfrac{2\sqrt{3}+2}{2\sqrt{3}-2}\Rightarrow\theta\approx1.309$
w is in the second quadrant
$\arg w=\pi-\theta=1.83$, to 3 significant figures

$\arg w$ is exactly $\dfrac{7\pi}{12}$. That would be an excellent answer to give, but an exact answer is not specified, so it is not essential. A calculator has been used here. Radians are not specified so degrees would also be acceptable.
$\arg w=105°$, exactly.

c $|w|^2=(2-2\sqrt{3})^2+(2+2\sqrt{3})^2$
$=4-8\sqrt{3}+12+4+8\sqrt{3}+12=32=16\times2$
$|w|=4\sqrt{2}$

d



A has coordinates $(1,\sqrt{3})$ and B has coordinates $(2-2\sqrt{3},2+2\sqrt{3})$. You use the formula $AB^2=(x_2-x_1)^2+(y_2-y_1)^2$ from Coordinate Geometry to calculate AB^2.

e $AB^2=(2-2\sqrt{3}-1)^2+(2+2\sqrt{3}-\sqrt{3})^2$
$=(1-2\sqrt{3})^2+(2+\sqrt{3})^2$
$=1-4\sqrt{3}+12+4+4\sqrt{3}+3$
$=20=4\times5$
$AB=2\sqrt{5}$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 22

Question:

The solutions of the equation $z^2 + 6z + 25 = 0$ are z_1 and z_2, where $0 < \arg z_1 < \pi$ and $-\pi < \arg z_2 < 0$.

a Express z_1 and z_2 in the form $a + \mathrm{i}b$, where a and b are integers.

b Show that $z_1^2 = -7 - 24\mathrm{i}$.

c Find $|z_1^2|$.

d Find $\arg\left(z_1^2\right)$.

e Show, on an Argand diagram, the points which represent the complex numbers z_1, z_2 and z_1^2.

Solution:

a $z^2+6z \quad =-25$

$z^2+6z+9=-25+9$

$(z+3)^2=-16$

$z=-3\pm 4\mathrm{i}$

$z_1=-3+4\mathrm{i}$, $z_2=-3-4\mathrm{i}$

$0<\arg z_1<\pi$ implies that z_1 is in the first or second quadrant . $-3+4\mathrm{i}$ is in the second quadrant, so this is z_1. The other solution is in the fourth quadrant, so that is z_2. You need to know which solution is z_1 before going on to parts (b), (c) and (d).

b $z_1^2=(-3+4\mathrm{i})^2=9-24\mathrm{i}-16$

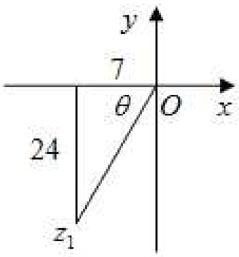
$\quad =-7-24\mathrm{i}$, as required

c $\left|z_1^2\right|^2=(-7)^2+(-24)^2=625$

$\left|z_1^2\right|=\sqrt{625}=25$

If you recognise 7, 24, 25 as a set of numbers satisfying the Pythagoras relation $a^2=b^2+c^2$, you can just write this answer down.

d



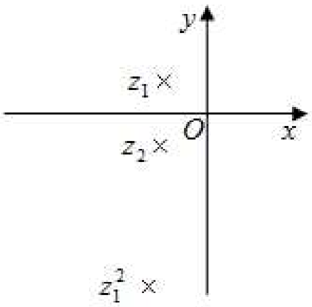
$\tan\theta=\dfrac{24}{7} \Rightarrow \theta\approx 1.287$

z_1 is in the fourth quadrant

$\arg z_1=-(\pi-\theta)=-1.85$, to 3 significant figures

The inequalities $0<\arg z_1<\pi$ and $-\pi<\arg z_2<0$ show that, in this question, the arguments are in radians.
Where no accuracy is specified in the question, it is reasonable to give your answer to 3 significant figures.

e



Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 23

Question:

$z = \sqrt{3} - \text{i}$. z^* is the complex conjugate of z.

a Show that $\frac{z}{z^*} = \frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}$.

b Find the value of $\left|\frac{z}{z^*}\right|$.

c Verify, for $z = \sqrt{3} - \text{i}$, that $\arg \frac{z}{z^*} = \arg z - \arg z^*$.

d Display z, z^* and $\frac{z}{z^*}$ on a single Argand diagram.

e Find a quadratic equation with roots z and z^* in the form $ax^2 + bx + c = 0$, where a, b and c are real constants to be found.

Solution:

a $z^* = \sqrt{3} + \mathrm{i}$

$$\frac{z}{z^*} = \frac{\sqrt{3}-\mathrm{i}}{\sqrt{3}+\mathrm{i}} \times \frac{\sqrt{3}-\mathrm{i}}{\sqrt{3}-\mathrm{i}} = \frac{(\sqrt{3}-\mathrm{i})^2}{(\sqrt{3})^2+1}$$

$$= \frac{(\sqrt{3})^2 - 2\sqrt{3}\mathrm{i} + \mathrm{i}^2}{3+1} = \frac{3-2\sqrt{3}\mathrm{i}-1}{4}$$

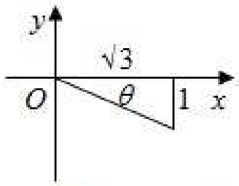
$$= \frac{2-2\sqrt{3}\mathrm{i}}{4} = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$$, as required

You multiply the numerator and the denominator by the conjugate complex of the denominator. The conjugate complex of $\sqrt{3}+\mathrm{i}$ is $\sqrt{3}-\mathrm{i}$, so the numerator becomes $(\sqrt{3}-\mathrm{i})^2$, which you can square using the formula $(a-b)^2 = a^2 - 2ab + b^2$.

b $$\left|\frac{z}{z^*}\right|^2 = \left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$$

$$\left|\frac{z}{z^*}\right| = 1$$

c



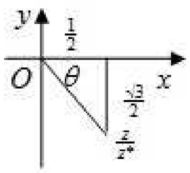
$$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$$

z is in the fourth quadrant

$$\arg z = -\frac{\pi}{6}$$

$$\arg z^* = \frac{\pi}{6}$$

You use $\arg z^* = -\arg z$ and $-\left(-\frac{\pi}{6}\right) = \frac{\pi}{6}$.



$$\tan\theta = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$$

$\frac{z}{z^*}$ is in the fourth quadrant

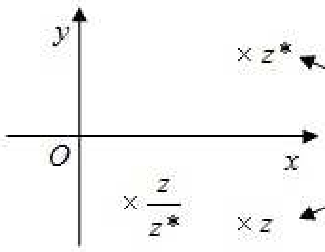
$$\arg\frac{z}{z^*} = -\frac{\pi}{3}$$

$$\arg z - \arg z^* = -\frac{\pi}{6} - \frac{\pi}{6} = -\frac{\pi}{3}$$

$= \arg\frac{z}{z^*}$, verifying the result

Verify means show that the equation is satisfied by the particular numbers in this question.
You should show that the equation is satisfied exactly and not use a calculator giving approximate results.

d



In the Argand diagram, you must place points representing conjugate complex numbers symmetrically about the real x-axis.

e

$$(x-z_1)(x-z_2)=(x-\sqrt{3}+\text{i})(x-\sqrt{3}+\text{i})$$
$$=(x-\sqrt{3})^2+1$$
$$=x^2-2\sqrt{3}x+3+1$$
$$=x^2-2\sqrt{3}x+4$$

The equation is $x^2-2\sqrt{3}x+4=0$.

If $x=\alpha$ and $x=\beta$ are the solutions of a quadratic equation, then the equation, after it has been factorised, must be $(x-\alpha)(x-\beta)=0$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 24

Question:

$z = \frac{1 + 7i}{4 + 3i}$.

a Find the modulus and argument of z.

b Write down the modulus and argument of z^*.

In an Argand diagram, the points A and B represent $1 + 7i$ and $4 + 3i$ respectively and O is the origin. The quadrilateral $OABC$ is a parallelogram.

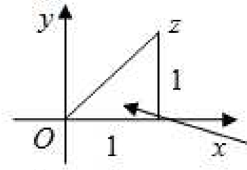
c Find the complex number represented by the point C.

d Calculate the area of the parallelogram.

Solution:

a $z = \dfrac{1+7\mathrm{i}}{4+3\mathrm{i}} \times \dfrac{4-3\mathrm{i}}{4-3\mathrm{i}} = \dfrac{4-3\mathrm{i}+28\mathrm{i}+21}{4^2+3^2}$

$= \dfrac{25+25\mathrm{i}}{25} = 1+\mathrm{i}$



$|z| = \sqrt{(1^2+1^2)} = \sqrt{2}$

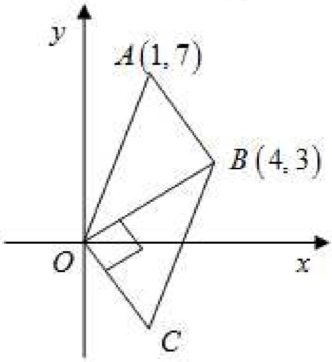
$\arg z = \dfrac{\pi}{4}$

You can see from the diagram that the argument is $45° = \dfrac{\pi}{4}$ and you need give no further working.

b $|z^*| = \sqrt{2}$, $\arg z^* = -\dfrac{\pi}{4}$

z^* is the symbol for the conjugate complex of z and you use the relations $|z^*| = |z|$ and $\arg z^* = -\arg z$ to write down the answers.

c



You are not asked to draw an Argand diagram in this question but you will certainly need to sketch one to sort out parts (c) and (d).

Let the complex number represented by the point C be w.
$OABC$ is a parallelogram. Therefore

$\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB}$

$1+7\mathrm{i}+w = 4+3\mathrm{i}$

$w = 3-4\mathrm{i}$

You use the representation of the addition of complex numbers in an Argand diagram. The diagonal OB of the parallelogram represents the addition of the two adjacent sides, OA and OC, of the parallelogram.

d $OB^2 = 4^2+3^2 = 25 \Rightarrow OB = 5$

$OC^2 = (-3)^2 + 4^2 = 25 \Rightarrow OC = 5$

The gradient of OB is given by $m = \dfrac{3}{4}$

The gradient of OC is given by $m' = -\dfrac{4}{3}$

$mm' = -1$ and, hence, OB is perpendicular to OC.
The area of the right-angled triangle OBC is given by

$$\text{area} = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2} \times 5 \times 5 = 12\frac{1}{2}$$

The area of the parallelogram is $2 \times 12\dfrac{1}{2} = 25$.

The diagonal of the parallelogram divides the parallelogram into two congruent triangles.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 25

Question:

Given that $\frac{z+2\mathrm{i}}{z-\lambda\mathrm{i}}=\mathrm{i}$, where λ is a positive, real constant,

a show that $z=\left(\frac{\lambda}{2}+1\right)+\mathrm{i}\left(\frac{\lambda}{2}-1\right)$.

Given also that $\tan(\arg z)=\frac{1}{2}$, calculate

b the value of λ,

c the value of $|z|^2$.

Solution:

a

$$\frac{z+2\mathrm{i}}{z-\lambda\mathrm{i}}=\mathrm{i}$$

$$z+2\mathrm{i}=\mathrm{i}(z-\lambda\mathrm{i})=\mathrm{i}z+\lambda$$

$$z(1-\mathrm{i})=\lambda-2\mathrm{i}$$

$$z=\frac{\lambda-2\mathrm{i}}{1-\mathrm{i}}\times\frac{1+\mathrm{i}}{1+\mathrm{i}}=\frac{\lambda+\lambda\mathrm{i}-2\mathrm{i}+2}{2}$$

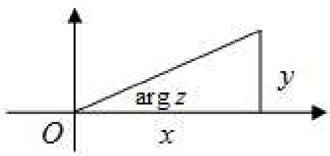
$$=\left(\frac{\lambda+2}{2}\right)+\mathrm{i}\left(\frac{\lambda-2}{2}\right)$$

$$=\left(\frac{\lambda}{2}+1\right)+\mathrm{i}\left(\frac{\lambda}{2}-1\right)\text{, as required}$$

You start this question by "making z the subject of the formula"; a method you learnt for GCSE.

b

$$\tan(\arg z)=\frac{\frac{\lambda}{2}-1}{\frac{\lambda}{2}+1}=\frac{\lambda-2}{\lambda+2}=\frac{1}{2}$$



If $z=x+\mathrm{i}y$, then $\tan(\arg z)=\frac{y}{x}$

$$2\lambda-4=\lambda+2 \quad\Rightarrow\lambda=6$$

Multiplying all terms in both the numerator and denominator by 2.

c Substitute $\lambda=6$ into the result of part (a).

$$z=\left(\frac{6}{2}+1\right)+\mathrm{i}\left(\frac{6}{2}-1\right)=4+2\mathrm{i}$$

$$|z|^2=4^2+2^2=20$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 26

Question:

The complex numbers $z_1 = 2 + 2\text{i}$ and $z_2 = 1 + 3\text{i}$ are represented on an Argand diagram by the points P and Q respectively.

a Display z_1 and z_2 on the same Argand diagram.

b Find the exact values of $|z_1|$, $|z_2|$ and the length of PQ.

Hence show that

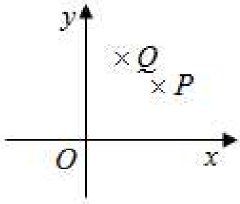
c ΔOPQ, where O is the origin, is right-angled.

Given that $OPQR$ is a rectangle in the Argand diagram,

d find the complex number z_3 represented by the point R.

Solution:

a



b $|z_1|^2 = 2^2 + 2^2 = 8 = 4 \times 2 \Rightarrow |z_1| = 2\sqrt{2}$

$|z_2|^2 = 1^2 + 3^2 = 10 \Rightarrow |z_2| = \sqrt{10}$

P has coordinates $(2, 2)$ and Q $(1, 3)$

$PQ^2 = (1-2)^2 + (3-2)^2 = (-1)^2 + 1^2 = 2$

$PQ = \sqrt{2}$

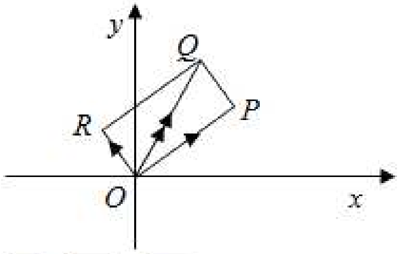
You use the formula $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ from Coordinate Geometry to calculate PQ^2.

c From (b), $OP = 2\sqrt{2}$ and $OQ = \sqrt{10}$.

$$OP^2 + PQ^2 = (2\sqrt{2})^2 + (\sqrt{2})^2$$
$$= 8 + 2 = 10$$
$$= OQ^2$$

By the converse of Pythagoras' Theorem, $\triangle OPQ$ is right-angled.

d



$\overline{OP} + \overline{OR} = \overline{OQ}$

$2 + 2\text{i} + z_3 = 1 + 3\text{i}$

$z_3 = -1 + \text{i}$

You use the representation of the addition of complex numbers in an Argand diagram. The diagonal OQ of the parallelogram represents the addition of the two adjacent sides, OP and OR, of the parallelogram. (A rectangle is a special case of a parallelogram.)

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 27

Question:

The complex number z is given by $z = (1 + 3\text{i})(p + q\text{i})$, where p and q are real and $p > 0$.

Given that $\arg z = \frac{\pi}{4}$,

a show that $p + 2q = 0$.

Given also that $|z| = 10\sqrt{2}$,

b find the value of p and the value of q.

c Write down the value of $\arg z^*$.

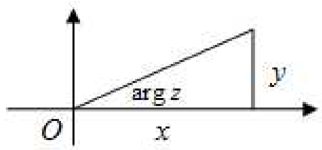
Solution:

a $z = (1+3\mathrm{i})(p+q\mathrm{i})$
$= p + q\mathrm{i} + 3p\mathrm{i} - 3q$
$= (p-3q) + (3p+q)\mathrm{i}$

$\arg z = \frac{\pi}{4}$, given

$\tan(\arg z) = \tan\frac{\pi}{4}$

$\frac{3p+q}{p-3q} = 1$



If $z = x + \mathrm{i}\,y$, then $\tan(\arg z) = \frac{y}{x}$.

$3p+q = p-3q \Rightarrow 2p+4q = 0$
$\Rightarrow p+2q = 0$, as required

b $|z|^2 = (p-3q)^2 + (3p+q)^2 = (10\sqrt{2})^2$
$p^2 - 6pq + 9q^2 + 9p^2 + 6pq + q^2 = 200$
$10p^2 + 10q^2 = 200$
$p^2 + q^2 = 20 \quad \ldots \ldots$ ❶
From the result of part (a)
$p = -2q \quad \ldots \ldots \ldots$ ❷
Substitute ❷ into ❶
$4q^2 + q^2 = 20 \Rightarrow 5q^2 = 20 \Rightarrow q^2 = 4$
$q = \pm 2$
$q = 2$ substituted into ❷ gives $p = -4$. As $p > 0$ is given in the question, this solution is rejected and $q = -2$ is the only answer.
$p = 4,\ q = -2$

❶ and ❷ are a pair of simultaneous equations, one of which is quadratic and the other linear. The method of solving these is given in Edexcel AS and A Level Modular Mathematics Core Mathematics 1, Chapter 3.

c $\arg z^* = -\frac{\pi}{4}$

You use $\arg z^* = -\arg z$ to write down this answer. You were given that $\arg z = \frac{\pi}{4}$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 28

Question:

The complex numbers z_1 and z_2 are given by $z_1 = 5 + \text{i}$, $z_2 = 2 - 3\text{i}$.

a Show points representing z_1 and z_2 on an Argand diagram.

b Find the modulus of $z_1 - z_2$.

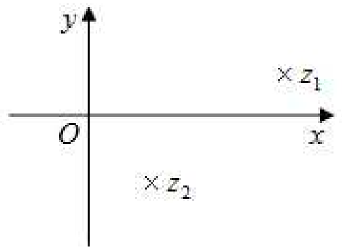
c Find the complex number $\frac{z_1}{z_2}$ in the form $a + \text{i}b$, where a and b are rational numbers.

d Hence find the argument of $\frac{z_1}{z_2}$, giving your answer in radians to three significant figures.

e Determine the values of the real constants p and q such that $\frac{p + \text{i}q + 3z_1}{p - \text{i}q + 3z_2} = 2\text{i}$.

Solution:

a



b $z_1 - z_2 = 5 + \mathrm{i} - (2 - 3\mathrm{i}) = 3 + 4\mathrm{i}$

$|z_1 - z_2|^2 = 3^2 + 4^2 = 25$

$|z_1 - z_2| = 5$

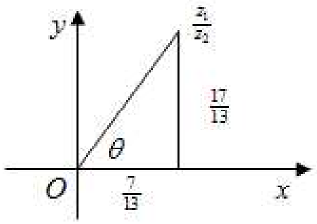
If you recognise the 3,4,5 "triangle", you can write the answer 5 down without further working.

c $\dfrac{z_1}{z_2} = \dfrac{5+\mathrm{i}}{2-3\mathrm{i}} \times \dfrac{2+3\mathrm{i}}{2+3\mathrm{i}} = \dfrac{10+15\mathrm{i}+2\mathrm{i}-3}{2^2+3^2}$

$= \dfrac{7+17\mathrm{i}}{13} = \dfrac{7}{13} + \dfrac{17}{13}\mathrm{i}$

The question asks you to put your answer in the form $a + \mathrm{i}b$, where a and b are rational numbers. Rational numbers are exact fractions and so $\frac{7}{13}$ and $\frac{17}{13}$ satisfy the conditions of the question. Approximate decimals would not be acceptable

d



$\tan\theta = \dfrac{\frac{17}{13}}{\frac{7}{13}} = \dfrac{17}{7} \Rightarrow \theta \approx 1.180$

$\dfrac{z_1}{z_2}$ is in the first quadrant

$\arg\dfrac{z_1}{z_2} = 1.18$, to 3 significant figures

e $p + \mathrm{i}q + 3z_1 = 2\mathrm{i}(p - \mathrm{i}q + 3z_2)$

$p + \mathrm{i}q + 15 + 3\mathrm{i} = 2p\mathrm{i} + 2q + 6\mathrm{i}(2 - 3\mathrm{i})$

$= 2p\mathrm{i} + 2q + 12\mathrm{i} + 18$

You find two simultaneous equations by equating the real and imaginary parts of the equation.

Equating real parts

$p + 15 = 2q + 18 \Rightarrow p - 2q = 3$ … ❶

Equating imaginary parts

$q + 3 = 2p + 12 \Rightarrow -2p + q = 9$ … ❷

❶ $\times 2$ $\qquad 2p - 4q = 6$ … ❸

❷ + ❸ $\qquad -3q = 15 \Rightarrow q = -5$

Substitute into ❶

$p + 10 = 3 \Rightarrow p = -7$

$p = -7, q = -5$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 29

Question:

$z = a + \mathrm{i}b$, where a and b are real and non-zero.

a Find z^2 and $\frac{1}{z}$ in terms of a and b, giving each answer in the form $x + \mathrm{i}y$, where x and y are real.

b Show that $|z^2| = a^2 + b^2$.

c Find $\tan(\arg z^2)$ and $\tan\left(\arg \frac{1}{z}\right)$, in terms of a and b.

On an Argand diagram the point P represents z^2 and the point Q represents $\frac{1}{z}$ and O the origin.

d Using your answer to **c**, or otherwise, show that if P, O and Q are collinear, then $3a^2 = b^2$.

Solution:

a $z^2 = (a + \mathrm{i}b)^2 = a^2 + 2ab\mathrm{i} - b^2$

$$= (a^2 - b^2) + 2ab\mathrm{i}$$

$$\frac{1}{z} = \frac{1}{a+\mathrm{i}b} \times \frac{a-\mathrm{i}b}{a-\mathrm{i}b} = \frac{a-\mathrm{i}b}{a^2+b^2}$$

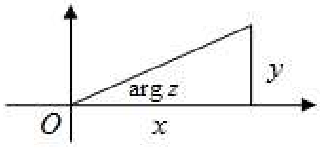
$$= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\mathrm{i}$$

b $|z^2|^2 = (a^2 - b^2)^2 + (2ab)^2$

$$= a^4 - 2a^2b^2 + b^4 + 4a^2b^2$$

$$= a^4 + 2a^2b^2 + b^4 = (a^2 + b^2)^2$$

Hence $|z^2| = a^2 + b^2$, as required.



c $\tan(\arg z^2) = \dfrac{2ab}{a^2 - b^2}$

$$\tan\left(\arg\frac{1}{z}\right) = \frac{-\dfrac{b}{a^2+b^2}}{\dfrac{a}{a^2+b^2}} = -\frac{b}{a}$$

If $z = x + \mathrm{i}y$, then $\tan(\arg z) = \dfrac{y}{x}$. You the use the answers in part (a).

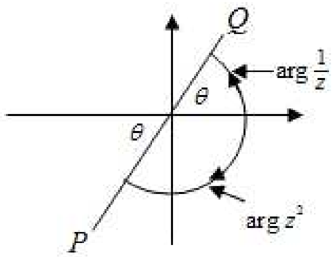
d If P, O and Q are in a straight line then $\tan(\arg z^2)$ and $\tan\left(\arg\dfrac{1}{z}\right)$ must be equal.

$$\frac{2ab}{a^2-b^2} = -\frac{b}{a}$$

$$2a^2\cancel{b} = -\cancel{b}(a^2 - b^2)$$

$$2a^2 = -a^2 + b^2$$

$$3a^2 = b^2\text{, as required}$$



If P and Q are in the same quadrant, this is obvious, but when they are in opposite quadrants this is not so clear. A possible case is shown above.

$$\tan(\arg z^2) = \tan(-(\pi - \theta)) = \tan(\theta - \pi)$$

$$= \tan\theta = \tan\left(\arg\frac{1}{z}\right)$$

$\tan(\theta - \pi) = \tan\theta$ because the function tan has period π. (This is in the C2 specification)
You would not be expected to explain this in an examination.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 30

Question:

Starting with $x = 1.5$, apply the Newton–Raphson procedure once to $f(x) = x^3 - 3$ to obtain a better approximation to the cube root of 3, giving your answer to three decimal places.

Solution:

$$f(x) = x^3 - 3$$

The cube root of x is the solution of $x^3 - 3 = 0$. 1.5 is the first approximation and you have to find a second approximation using the Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

$$f'(x) = 3x^2$$

$$f(1.5) = 1.5^3 - 3 = 0.375$$

$$f'(1.5) = 3 \times 1.5^2 = 6.75$$

$$x = 1.5 - \frac{f(1.5)}{f'(1.5)}$$

It is a good idea to write down the Newton-Raphson formula and adapt it to your solution. The examiner will then know what you are doing.

$$= 1.5 - \frac{0.375}{6.75}$$

$= 1.444$, to 3 decimal places

Correct the answer to the number of decimal places you are asked for.

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Review Exercise
Exercise A, Question 31

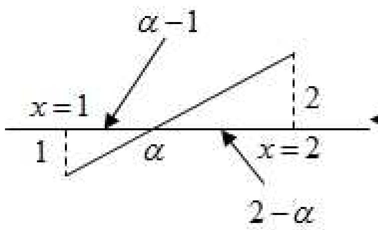
Question:

$f(x) = 2^x + x - 4$. The equation $f(x) = 0$ has a root α in the interval [1, 2]. Use linear interpolation on the values at the end points of this interval to find an approximation to α.

Solution:

$$f(x) = 2^x + x - 4$$
$$f(1) = 2^1 + 1 - 4 = -1$$
$$f(2) = 2^2 + 2 - 4 = 2$$

The first stage of a linear interpolation is to evaluate the function at both ends of the interval.



A diagram helps you to see what is going on and, as you are going to use similar triangles, to see which sides in one triangle correspond to which sides in the other triangle.

By similar triangles

$$\frac{\alpha - 1}{1} = \frac{2 - \alpha}{2}$$

Solve the equation to find α.

$$2\alpha - 2 = 2 - \alpha$$
$$3\alpha = 4$$
$$\alpha = 1\frac{1}{3}$$

This is an exact answer. There is no need to correct to a given number of decimal places as you have not been asked to do this.

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Review Exercise
Exercise A, Question 32

Question:

Given that the equation $x^3 - x - 1 = 0$ has a root near 1.3, apply the Newton–Raphson procedure once to $\mathrm{f}(x) = x^3 - x - 1$ to obtain a better approximation to this root, giving your answer to three decimal places.

Solution:

Let $\mathrm{f}(x) = x^3 - x - 1$

$\mathrm{f}'(x) = 3x^2 - 1$

$\mathrm{f}(1.3) = -0.103$

$\mathrm{f}'(1.3) = 4.07$

$$x = 1.3 - \frac{\mathrm{f}(1.3)}{\mathrm{f}'(1.3)}$$

$$= 1.3 + \frac{0.103}{4.07}$$

$= 1.325307\ \ldots$

≈ 1.325

Remember to correct your answer to the number of decimal places asked for in the question.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 33

Question:

$f(x) = x^3 - 12x + 7$.

a Use differentiation to find $f'(x)$.

The equation $f(x) = 0$ has a root α in the interval $\frac{1}{2} < x < 1$.

b Taking $x = \frac{1}{2}$ as a first approximation to α, use the Newton–Raphson procedure twice to obtain two further approximations to α. Give your final answer to four decimal places.

Solution:

a $f'(x) = 3x^2 - 12$

b $x_1 = 0.5$

0.5 $\left(\text{or } \frac{1}{2}\right)$ is the first approximation, which you are given. You have to find two more approximations. It is useful to call the first approximation x_1, the second x_2, the third x_3, etc. This helps you keep track of where you are in a long calculation.

$f(0.5) = 0.5^3 - 12 \times 0.5 + 7 = 1.125$

$f'(0.5) = 3 \times 0.5^2 - 12 = -11.25$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.5 - \frac{1.125}{-11.25} = 0.5 + 0.1$$

$$= 0.6$$

The signs need care here. Missing that "minus a minus is a plus" is a major source of error, even at A level!

$f(0.6) = 0.6^3 - 12 \times 0.6 + 7 = 0.016$

$f'(0.6) = 3 \times 0.6^2 - 12 = -10.92$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.6 - \frac{0.016}{-10.92} = 0.6 + 0.001465 \ldots$$

$= 0.6015$, to 4 decimal places

Remember to correct your answer to the number of decimal places asked for in the question.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 34

Question:

The equation $\sin x = \frac{1}{2}x$ has a root in the interval [1.8, 2]. Use linear interpolation once on the interval [1.8, 2] to find an estimate of the root, giving your answer to two decimal places.

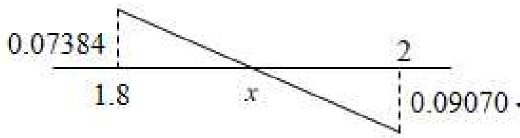
Solution:

$$f(x) = \sin x - \frac{1}{2}x$$

In this question you have not been given $f(x)$ and have to choose a function yourself. To choose $f(x) = \sin x - \frac{1}{2}x$ is sensible as $f(x) = 0$ is obviously equivalent to the equation $\sin x = \frac{1}{2}x$, which you are asked to solve.

$$f(1.8) = 0.07384\ \ldots$$
$$f(2) = -0.09070\ \ldots$$

Remember to work in radians. The final answer has to be given to 2 decimal places and you must work to sufficient accuracy to achieve this. 5 decimal places will certainly be enough to achieve this but there is no harm in giving more decimal places.



It is a common error to use -0.09070 instead of 0.09070 here. $f(2)$ is negative but the number is the length of a side in the diagram and lengths have to be positive.

By similar triangles

$$\frac{x-1.8}{0.07384} = \frac{2-x}{0.09070}$$

$$0.09070x - 0.16326 = 0.14768 - 0.07384x$$

$$0.16454x = 0.31094$$

The numbers are quite difficult here. Use your calculator to help you "cross multiply" and collect together correctly.

$$x = \frac{0.31094}{0.16454} \approx 1.89, \text{ to 2 decimal places.}$$

Your answer must be corrected to 2 decimal places – the accuracy specified in the question.

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Review Exercise
Exercise A, Question 35

Question:

$f(x) = x^4 + 3x^3 - 4x - 5$. The equation $f(x) = 0$ has a root between $x = 1.2$ and $x = 1.6$. Starting with the interval [1.2, 1.6], use interval bisection three times to obtain an interval of width 0.05 which contains this root.

Solution:

The mid-point of the interval $[1.2, 1.6]$ is

$$\frac{1.2+1.6}{2}=1.4$$

You start interval bisection by dividing the interval into two equal parts by finding the mid-point of an interval.

$$f(1.2)=-2.5424<0$$
$$f(1.4)=1.4736>0$$
$$(f(1.6)=7.4416)$$

It is not always necessary to calculate the values at both ends and the mid-point. In this case you already have a sign change between $x=1.2$ and $x=1.4$ and, so it is not necessary to calculate the value of $f(1.6)$.

There is a sign change between $x=1.2$ and $x=1.6$.
Hence, the root lies in the interval $(1.2, 1.4)$.

The mid-point of the interval $[1.2, 1.4]$ is

$$\frac{1.2+1.4}{2}=1.3$$

$$f(1.3)=-0.7529<0$$
$f(1.4)=1.4736>0$, from above.

You calculated $f(1.4)$ earlier and there is no need to calculate it again.

There is a sign change between $x=1.3$ and $x=1.4$.
Hence, the root lies in the interval $(1.3, 1.4)$.

The mid-point of the interval $[1.3, 1.4]$ is

$$\frac{1.3+1.4}{2}=1.35$$

$f(1.35)=0.30263>0$, from above.
$$f(1.3)=-0.7529<0$$

There is a sign change between $x=1.3$ and $x=1.35$.
Hence, the root lies in the interval $(1.3, 1.35)$.

$1.35-1.3=0.05$ and so this interval satisfies the requirements of the question.

Quartic equations can be solved exactly. You may have access to a computer package or advanced calculator which can do this. $x=1.336\,20$ is accurate to 5 decimal places, which confirms the result of your calculation.

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Review Exercise
Exercise A, Question 36

Question:

$\mathrm{f}(x) = 3\tan\left(\frac{x}{2}\right) - x - 1, \ -\pi < x < \pi.$

Given that $\mathrm{f}(x) = 0$ has a root between 1 and 2, use linear interpolation once on the interval [1, 2] to find an approximation to this root. Give your answer to two decimal places.

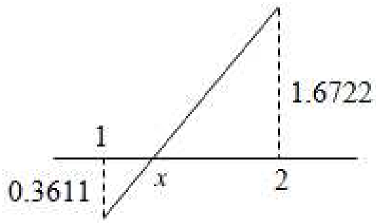
Solution:

$$\mathrm{f}(x) = 3\tan\left(\frac{x}{2}\right) - x - 1$$

Unless it is clearly stated otherwise, all questions on this topic require you to work in radians. Make sure your calculator is in the correct mode.

$$\mathrm{f}(1) = -0.3611$$
$$\mathrm{f}(2) = 1.6722$$

The final answer must be to 2 decimal places. To achieve this you must work to at least one more decimal place and it's safer to work to more. 4 decimal places will certainly be enough here.



By similar triangles

$$\frac{x-1}{0.3611} = \frac{2-x}{1.6722}$$

$$1.6722x - 1.6722 = 0.7222 - 0.3611x$$

$$2.0333x = 2.39442$$

$$x = \frac{2.39442}{2.0333} \approx 1.1776$$

Correct your approximation to x to 2 decimal places.

$x \approx 1.18$, to 2 decimal places

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Review Exercise
Exercise A, Question 37

Question:

$f(x) = 3^x - x - 6$.

a Show that $f(x) = 0$ has a root α between $x = 1$ and $x = 2$.

b Starting with the interval [1, 2], use interval bisection three times to find an interval of width 0.125 which contains α.

Solution:

a

$$f(x) = 3^x - x - 6$$
$$f(1) = 3 - 1 - 6 = -4 < 0$$
$$f(2) = 9 - 2 - 6 = 1 > 0$$

There is a sign change between $x = 1$ and $x = 2$.
Hence the function $f(x)$ has a root α between $x = 1$ and $x = 2$.

When you are asked to "show that", or "prove that" a result is true, you should give a conclusion to your argument. It is always safe to base the wording of your conclusion on the wording of the question, as has been done here.

b

$$\frac{1+2}{2} = 1.5$$

At each stage of an interval bisection question, you begin by dividing the interval into two equal parts by finding its mid-point.

$$f(1.5) = -2.3038\ \ldots < 0$$
$$f(2) = 1 > 0\text{, from above.}$$

You calculated $f(1)$ and $f(2)$ in part (a) of the question and there is no need to calculate them again in part (b).

There is a sign change between $x = 1.5$ and $x = 2$.
Hence $\alpha \in (1.5, 2)$.

$$\frac{1.5+2}{2} = 1.75$$

$$f(1.75) = -0.9114 < 0$$
$$f(2) = 1 > 0\text{, from above.}$$

There is a sign change between $x = 1.75$ and $x = 2$.
Hence $\alpha \in (1.75, 2)$.

$$\frac{1.75+2}{2} = 1.875$$

$$f(1.875) = -0.0298 < 0$$
$$f(2) = 1 > 0\text{, from above.}$$

There is a sign change between $x = 1.875$ and $x = 2$.
Hence $\alpha \in (1.875, 2)$.

$2 - 1.875 = 0.125$ and so this interval satisfies the conditions of the question.

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Review Exercise
Exercise A, Question 38

Question:

Given that x is measured in radians and $f(x) = \sin x - 0.4x$,

a find the values of f(2) and f(2.5) and deduce that the equation $f(x) = 0$ has a root α in the interval [2, 2.5],

b use linear interpolation once on the interval [2, 2.5] to estimate the value of α, giving your answer to two decimal places.

Solution:

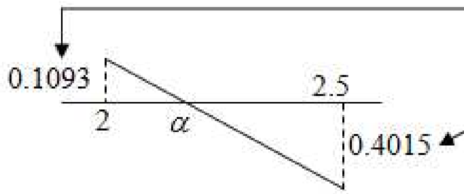
a

$$f(x) = \sin x - 0.4x$$
$$f(2) = 0.10929\ \ldots > 0$$
$$f(2.5) = -0.40152\ \ldots < 0$$

There is a sign change between $x = 2$ and $x = 2.5$.

Hence the equation $f(x) = 0$ has a root α in the interval $[2, 2.5]$.

b



You have calculated the values of $f(x)$ at the end points of the interval in part (a) and these values can be used in part (b).

By similar triangles

$$\frac{\alpha - 2}{0.1093} = \frac{2.5 - \alpha}{0.4015}$$

The answer needs to be given to 2 decimal places; that will be 3 significant figures. It will be sufficient to work to 4 significant figures here. There would be no harm in using more significant figures but if you only worked to 3 significant figures the last figure might be inaccurate.

$$0.4015\alpha - 0.8030 = 0.2733 - 0.1093\alpha$$
$$0.5108\alpha = 1.0765$$
$$\alpha \approx 2.11, \text{ to 2 decimal places.}$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 39

Question:

$f(x) = \tan x + 1 - 4x, \; -\frac{\pi}{2} < x < \frac{\pi}{2}.$

a Show that $f(x) = 0$ has a root α in the interval [1.42, 1.44].

b Use linear interpolation once on the interval [1.42, 1.44] to find an estimate of α, giving your answer to three decimal places.

Solution:

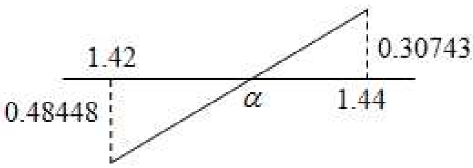
a

$$f(x) = \tan x + 1 - 4x^2$$
$$f(1.42) \approx -0.48448 < 0$$
$$f(1.44) \approx 0.30743 > 0$$

There is a sign change between $x = 1.42$ and $x = 1.44$.
Hence the equation $f(x) = 0$ has a root α in the interval $[1.42, 1.44]$.

To show a change of sign, you only need to calculate the values of the function to one significant figure. However later in the question you are asked to give your answer to 3 decimal places (which will be 4 significant figures). It is sensible to work out and write down at least 5 significant figures here. You do not want to carry out or write out the calculations twice. It often pays to read quickly through a question before you start it.

b



By similar triangles

$$\frac{\alpha - 1.42}{0.48448} = \frac{1.44 - \alpha}{0.30743}$$

$$(0.30743 + 0.48448)\alpha$$
$$= 1.44 \times 0.48448 + 1.42 \times 0.30743$$

$$0.7919\alpha = 1.1342018$$
$$\alpha \approx 1.432, \text{ to 3 decimal places.}$$

Solutionbank FP1
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Review Exercise
Exercise A, Question 40

Question:

$f(x) = \cos\sqrt{x} - x$

a Show that $f(x) = 0$ has a root α in the interval [0.5, 1].

b Use linear interpolation on the interval [0.5, 1] to obtain an approximation to α. Give your answer to two decimal places.

c By considering the change of sign of $f(x)$ over an appropriate interval, show that your answer to **b** is accurate to two decimal places.

Solution:

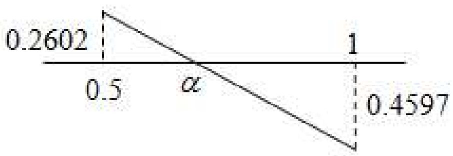
a

$$f(x) = \cos\sqrt{x} - x$$
$$f(0.5) = 0.2602 > 0$$
$$f(1) = -0.4597 < 0$$

In this topic, angles are measured in radians, unless otherwise stated.

There is a sign change between $x = 0.5$ and $x = 1$.
Hence the equation $f(x) = 0$ has a root α in the interval $[0.5, 1]$.

b



By similar triangles

$$\frac{\alpha - 0.5}{0.2602} = \frac{1-\alpha}{0.4597}$$

$$0.4597\alpha - 0.2299 = 0.2602 - 0.2602\alpha$$
$$0.7199\alpha = 0.4901$$
$$\alpha \approx 0.68\text{, to 2 decimal places}$$

c

$$f(0.675) = 0.006\,06 \ldots > 0$$
$$f(0.685) = -0.008\,38 \ldots < 0$$

There is a change of sign and, hence, $\alpha \in (0.675, 0.685)$.
Hence $\alpha = 0.68$ is accurate to 2 decimal places.

If 0.68 is accurate to 2 decimal places then α must lie in the interval $0.675 \leqslant \alpha < 0.685$. Any number in this interval rounded to two decimal places is 0.68. You evaluate $f(x)$ at the end points of this interval and, if there is a change of sign, you know that α lies in the interval and you can deduce that 0.68 is **accurate** to 2 decimal places.

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Review Exercise
Exercise A, Question 41

Question:

$f(x) = 2^x - x^2 - 1$

The equation $f(x) = 0$ has a root α between $x = 4.256$ and $x = 4.26$.

a Starting with the interval [4.256, 4.26] use interval bisection three times to find an interval of width 5×10^{-4} which contains α.

b Write down the value of α, correct to three decimal places.

Solution:

a $$\frac{4.256+4.26}{2} = 4.258$$

$f(4.256) = -0.0069 \ldots < 0$
$f(4.258) = 0.0025 \ldots > 0$

As you already have a change of sign, there is no need to calculate $f(4.26)$.

There is a sign change between $x = 4.256$ and $x = 4.258$.
Hence $\alpha \in [4.256, 4.258]$.

$$\frac{4.256+4.258}{2} = 4.257$$

$f(4.257) = -0.0021 \ldots < 0$
$f(4.258) = 0.0025 \ldots > 0$, from above

There is a sign change between $x = 4.257$ and $x = 4.258$.
Hence $\alpha \in [4.257, 4.258]$.

$$\frac{4.257+4.258}{2} = 4.2575$$

$f(4.257) = -0.0021 \ldots < 0$, from above
$f(4.2575) = 0.00018 \ldots > 0$

There is a sign change between $x = 4.257$ and $x = 4.2575$.
Hence $\alpha \in [4.257, 4.2575]$.

$4.2575 - 4.257 = 0.0005$, which is the same as 5×10^{-4}, and so the interval $[4.257, 4.2575]$ satisfies the conditions in the question. The open interval $(4.257, 4.2575)$ would also be correct.

b As $\alpha \in [4.257, 4.2575]$, then $\alpha = 4.257$ is accurate to 3 decimal places.

Any number in the interval $[4.257, 4.2575]$ rounded to 3 decimal places would be 4.257. Accurately $\alpha = 4.257\,4619 \ldots$ which is 4.257, to 3 decimal places.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 42

Question:

$f(x) = 2x^2 + \frac{1}{x} - 3$

The equation $f(x) = 0$ has a root α in the interval $0.3 < x < 0.5$.

a Use linear interpolation once on the interval $0.3 < x < 0.5$ to find an approximation to α. Give your answer to three decimal places.

b Find $f'(x)$.

c Taking 0.4 as an approximation to α, use the Newton–Raphson procedure once to find another approximation to α.

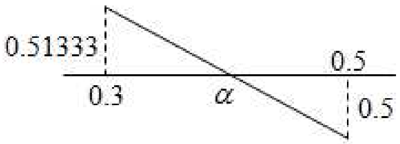
Solution:

a

$$f(x) = 2x^2 + \frac{1}{x} - 3$$

$$f(0.3) = 0.513\,33\ \ldots > 0$$

$$f(0.5) = -0.5 < 0$$



By similar triangles

$$\frac{\alpha - 0.3}{0.51333} = \frac{0.5 - \alpha}{0.5}$$

$$(0.5 + 0.51333)\alpha = 0.5 \times 0.51333 + 0.3 \times 0.5$$

$$1.01333\alpha = 0.4066$$

$\alpha \approx 0.401$, to 3 decimal places.

b $f'(x) = 4x - \frac{1}{x^2}$

$\frac{1}{x} = x^{-1}$ and the rule for differentiation $\frac{d}{dx}(x^n) = nx^{n-1}$ gives $\frac{d}{dx}(x^{-1}) = -1 \times x^{-2} = -\frac{1}{x^2}$

c

$$f(0.4) = -0.18$$

$$f(0.4) = -4.65$$

$$\alpha = 0.4 - \frac{f(0.4)}{f'(0.4)}$$

$$= 0.4 - \frac{0.18}{4.65}$$

$$\alpha \approx 0.361$$

No accuracy has been specified in the question. Giving the answer to 2 or 3 significant figures is reasonable.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 43

Question:

$f(x) = 0.25x - 2 + 4 \sin\sqrt{x}$.

a Show that the equation $f(x) = 0$ has a root α between $x = 0.24$ and $x = 0.28$.

b Starting with the interval [0.24, 0.28], use interval bisection three times to find an interval of width 0.005 which contains α

Solution:

a

$$f(x) = 0.25x - 2 + 4\sin\sqrt{x}$$

Remember to carry out the calculations in radian mode.

$$f(0.24) \approx -0.06 < 0$$

$$f(0.28) \approx 0.09 > 0$$

In a question where you only have to consider sign changes, you need only work to one significant figure. The solution shown here gives the minimum of working. You can, of course, show more decimal places if you wish.

There is a sign change between $x = 0.24$ and $x = 0.28$.

Hence the equation $f(x) = 0$ has a root α between $x = 0.24$ and $x = 0.28$.

b

$$\frac{0.24 + 0.28}{2} = 0.26$$

$$f(0.26) \approx 0.02 > 0$$

$$f(0.24) \approx -0.06 < 0, \text{ from above}$$

There is a sign change between $x = 0.24$ and $x = 0.26$.

Hence $\alpha \in [0.24, 0.26]$.

$$\frac{0.24 + 0.26}{2} = 0.25$$

$$f(0.25) \approx -0.02 < 0$$

$$f(0.26) \approx 0.02 > 0, \text{ from above}$$

There is a sign change between $x = 0.25$ and $x = 0.26$.

Hence $\alpha \in [0.25, 0.26]$.

$$\frac{0.25 + 0.26}{2} = 0.255$$

$$f(0.255) \approx -0.001 < 0$$

$$f(0.26) \approx 0.02 > 0, \text{ from above}$$

There is a sign change between $x = 0.255$ and $x = 0.26$.

Hence $\alpha \in [0.255, 0.26]$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 44

Question:

$f(x) = x^3 + 8x - 19$.

a Show that the equation $f(x) = 0$ has only one real root.

b Show that the real root of $f(x) = 0$ lies between 1 and 2.

c Obtain an approximation to the real root of $f(x) = 0$ by performing two applications of the Newton–Raphson procedure to $f(x)$, using $x = 2$ as the first approximation. Give your answer to three decimal places.

d By considering the change of sign of $f(x)$ over an appropriate interval, show that your answer to **c** is accurate to three decimal places.

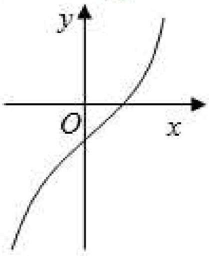
Solution:

a $\quad \text{f}'(x) = 3x^2 + 8$

As, for all x, $x^2 \geqslant 0$, $\text{f}'(x) \geqslant 8 > 0$ for all x.

As the derivative of $\text{f}(x)$ is always positive,

$\text{f}(x)$ is always increasing.



Drawing a sketch diagram helps you to see what is going on. If the function is always increasing, after crossing the x-axis it can never turn round and cross the axis again.

As $\text{f}(x)$ is always increasing it can only cross the x-axis once, as shown in the sketch and, hence, the equation $\text{f}(x) = 0$ has only one real root.

b $\quad \text{f}(1) = -10 < 0$

$\text{f}(2) = 5 > 0$

There is a sign change between $x = 1$ and $x = 2$.

Hence the real root of $\text{f}(x) = 0$ lies between $x = 1$ and $x = 2$.

You should give a conclusion to this part of the question. You can word the conclusion by modelling it upon the wording in the question.

c $\quad x_1 = 2$

$\text{f}(2) = 20$

$\text{f}'(2) = 5$

$$x_2 = 2 - \frac{\text{f}(2)}{\text{f}'(2)} = 2 - \frac{5}{20} = 1.75$$

This is the Newton-Raphson formula $x_{n+1} = x_n - \dfrac{\text{f}(x_n)}{\text{f}'(x_n)}$ with the values that apply in this question.

$\text{f}(1.75) = 0.359375$

$\text{f}'(1.75) = 17.1875$

$$x_3 = 1.75 - \frac{\text{f}(1.75)}{\text{f}'(1.75)} = 1.75 - \frac{0.359387}{17.1975}$$

≈ 1.729, to 3 decimal places

d $\quad \text{f}(1.7285) \approx -0.0077 < 0$

$\text{f}(1.7295) \approx 0.0092 > 0$

If 1.729 is accurate to 3 decimal places then α must lie in the interval $1.7285 \leqslant \alpha < 1.7295$. Any number in this interval rounded to 3 decimal places is 1.729. You evaluate $\text{f}(x)$ at the end points of this interval and, if there is a change of sign, you know that the root lies in the interval your answer is correct to 3 decimal places.

There is a change of sign between $x = 1.7285$ and $x = 1.7295$. Hence the root of the equation lies in the interval $(1.7285, 1.7295)$.

It follows that the root is 1.729 correct to 3 decimal places.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 45

Question:

$f(x) = x^3 - 3x - 1$

The equation $f(x) = 0$ has a root α in the interval $[-2, -1]$.

a Use linear interpolation on the values at the ends of the interval $[-2, -1]$ to obtain an approximation to α.

The equation $f(x) = 0$ has a root β in the interval $[-1, 0]$.

b Taking $x = -0.5$ as a first approximation to β, use the Newton–Raphson procedure once to obtain a second approximation to β.

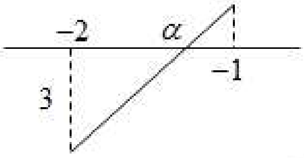
The equation $f(x) = 0$ has a root γ in the interval [1.8, 1.9].

c Starting with the interval [1.8, 1.9] use interval bisection twice to find an interval of width 0.025 which contains γ.

Solution:

a $f(-1)=(-1)^3-3(-1)-1=-1+3-1=1$

$f(-2)=(-2)^3-3(-2)-1=-8+6-1=-3$



$$\frac{\alpha-(-2)}{3}=\frac{-1-\alpha}{1}$$

$$\alpha+2=-3-3\alpha$$

$$4\alpha=-5$$

$$\alpha\approx-1.25$$

Finding distances on the negative x-axis can be difficult. The distance is the positive difference between the coordinates, so you must subtract the coordinates and, as $\alpha-(-2)=\alpha+2$, this will be positive when α is between -1 and -2.

b

$$f'(x)=3x^2-3$$

$$f(-0.5)=0.375$$

$$f'(-0.5)=-2.25$$

$$\beta=-0.5-\frac{f(-0.5)}{f'(-0.5)}=-0.5-\frac{0.375}{-2.25}$$

$$\beta\approx-0.33$$

This expression evaluates as exactly $-\frac{1}{3}$ but as this is an estimate of β, and not an exact value of β, it is sensible to give the answer to 2 decimal places.

c

$$\frac{1.8+1.9}{2}=1.85$$

$$f(1.8)=-0.568<0$$

$$f(1.85)=-0.218\ \ldots<0$$

$$f(1.9)=0.159>0$$

There is a sign change between $x=1.85$ and $x=1.9$.

Hence $\gamma\in(1.85,1.9)$.

$$\frac{1.85+1.9}{2}=1.875$$

$$f(1.875)\approx-0.0332<0$$

$f(1.9)=0.159>0$, as above

There is a sign change between $x=1.875$ and $x=1.9$.

Hence $\gamma\in(1.875,1.9)$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

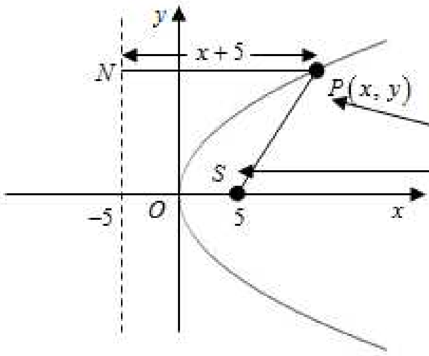
Review Exercise
Exercise A, Question 46

Question:

A point P with coordinates (x, y) moves so that its distance from the point $(5, 0)$ is equal to its distance from the line with equation $x = -5$.

Prove that the locus of P has an equation of the form $y^2 = 4ax$, stating the value of a.

Solution:



A diagram helps you see and understand what is going on. You should label important points. If letters are not given in the question, you can make your own up. Putting them on a diagram makes your method clear to the examiner.

By the definition of a parabola

$$SP = PN$$

$$SP^2 = PN^2$$

Using the formula $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$, it is often easier to work with the distances squared rather than with distances.

$S(5, 0)$, $P(x, y)$

$$SP^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$= (x-5)^2 + y^2$$

$$PN = x + 5$$

Along PN, the distance from P to the y-axis is x, and it is a further distance 5 from the y-axis to N.

$$SP^2 = PN^2$$

$$(x-5)^2 + y^2 = (x+5)^2$$

$$\cancel{x^2} - 10x + \cancel{25} + y^2 = \cancel{x^2} + 10x + \cancel{25}$$

$$y^2 = 20x$$

Multiply out the brackets using $(a+b)^2 = a^2 + 2ab + b^2$. Then "cancel" the equal terms on both sides of the equation.

Comparing with $y^2 = 4ax$, this is the required form with $a = 5$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 47

Question:

A parabola C has equation $y^2 = 16x$. The point S is the focus of the parabola.

a Write down the coordinates of S.

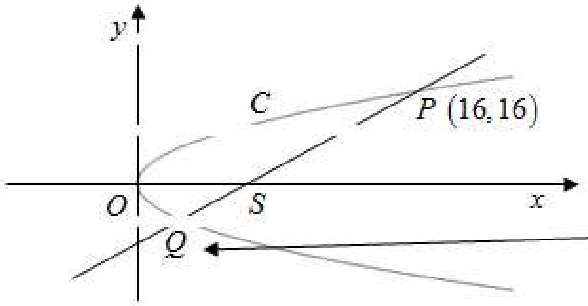
The point P with coordinates (16, 16) lies on C.

b Find an equation of the line SP, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

The line SP intersects C at the point Q, where P and Q are distinct points.

c Find the coordinates of Q.

Solution:



You should mark on your diagram any points given in the question. Here mark S, P and Q. Diagrams often help you check your working. Here, for example, it is obvious from the diagram that Q must have a negative y-coordinate. If you got $y = 4$ (a mistake it is easy to make), you would know you were wrong and look for an error in your working.

a $S(4, 0)$

The focus of the parabola with equation $y^2 = 4ax$ has coordinates $(a, 0)$. Here $a = 4$. The question asks you to write down the answer, so you do not have to show working.

b Using $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$ with $(x_1, y_1) = (4, 0)$ and $(x_2, y_2) = (16, 16)$, an equation of SP is

$$\frac{y-0}{16-0} = \frac{x-4}{16-4}$$

$$\frac{y}{16} = \frac{x-4}{12}$$

$$\cancel{12}^3 y = \cancel{16}^4 (x-4)$$

$$3y = 4x - 16$$

$$4x - 3y - 16 = 0$$

Methods for finding the equation of a straight line are given in Chapter 5 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 1. You can use any correct method for finding the line.

c From (b)

$$x = \frac{3y+16}{4}$$

Substitute for x in $y^2 = 16x$

$$y^2 = \cancel{16}^4 \left(\frac{3y+16}{\cancel{4}}\right) = 12y + 64$$

$$y^2 - 12y - 64 = 0.$$

$$(y-16)(y+4) = 0$$

$y = 16$ corresponds to the point P.

To find Q you solve the simultaneous equations $4x - 3y - 16 = 0$ and $y^2 = 16x$. The method of using substitution, when one equation is linear and the other is quadratic, is given in Chapter 3 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 1.

For Q, $y = -4$

$$x = \frac{3 \times -4 + 16}{4} = \frac{4}{4} = 1$$

The coordinates of Q are $(1, -4)$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 48

Question:

The curve C has equations $x = 3t^2$, $y = 6t$.

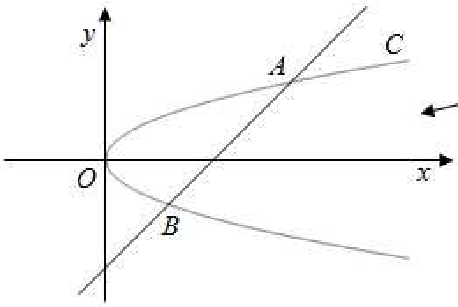
a Sketch the graph of the curve C.

The curve C intersects the line with equation $y = x - 72$ at the points A and B.

b Find the length AB, giving your answer as a surd in its simplest form.

Solution:

a



You have to recognise that $x = 3t^2$, $y = 6t$ is a parabola and draw it passing through the origin with the correct orientation.

b For the intersections, substitute $x = 3t^2$, $y = 6t$ into $y = x - 72$

$$6t = 3t^2 - 72$$
$$3t^2 - 6t - 72 = 0$$
$$(\div 3) \quad t^2 - 2t - 24 = 0$$
$$(t-6)(t+4) = 0$$
$$t = 6, -4$$

$(3t^2, 6t)$ is a general point on the parabola. The points A and B must be of this form and, if they also lie on the line with equation $y = x - 72$, the points on the parabola must also satisfy the equation of the line.

For A, say, $t = 6$

$$x = 3t^2 = 108, \; y = 6t = 36$$

For B, say, $t = -4$

$$x = 3t^2 = 3 \times (-4)^2 = 48, \; y = 6t = -24$$

The question does not tell you which point is A and which point is B but, as you are only asked to find the distance between them, it does not matter which is which and you can make your own choice.

Using $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$$AB^2 = (108 - 48)^2 + (36 - (-24))^2$$
$$= 60^2 + 60^2 = 2 \times 60^2$$
$$AB = \sqrt{(2 \times 60^2)} = 60\sqrt{2}$$

You are asked to give your answer as a surd in its simplest form. 84.85 is not acceptable as it is not a surd and $\sqrt{7200}$ is not the simplest form. A surd in its simplest form contains the square root of the smallest possible single number.

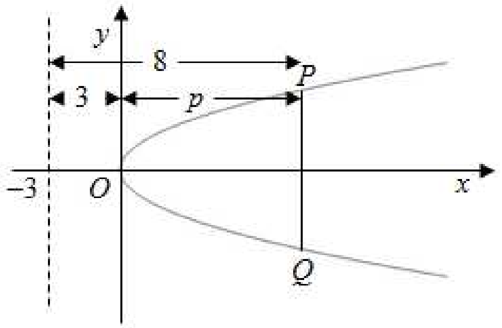
Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 49

Question:

A parabola C has equation $y^2 = 12x$. The points P and Q both lie on the parabola and are both at a distance 8 from the directrix of the parabola. Find the length PQ, giving your answer in surd form.

Solution:



The directrix of $y^2 = 4ax$ is $x = -a$. Comparison of $y^2 = 4ax$ with $y^2 = 12x$ shows that, in this question, $a = 3$.

The equation of the directrix is $x = -3$.
If the x-coordinate of P is p,

$$p + 3 = 8 \Rightarrow p = 5$$

The y-coordinate of P is given by

$$y^2 = 12x = 60 \Rightarrow y = \sqrt{60}$$

By symmetry, the coordinates of Q are $(5, -\sqrt{60})$

P is vertically above Q and the distance from P to Q is twice the y–coordinate of P.

$$PQ = 2\sqrt{60} = 4\sqrt{15}$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 50

Question:

The point $P(2, 8)$ lies on the parabola C with equation $y^2 = 4ax$. Find

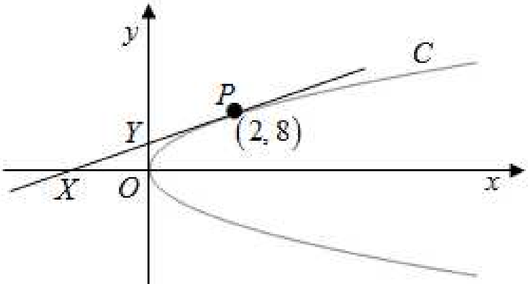
a the value of a,

b an equation of the tangent to C at P.

The tangent to C at P cuts the x-axis at the point X and the y-axis at the point Y.

c Find the exact area of the triangle OXY.

Solution:



a Substitute $(2, 8)$ into $y^2 = 4ax$.

$64 = 4a \times 2 = 8a \Rightarrow a = \frac{64}{8} = 8$

b $y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2} \times 2a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$

Using $\frac{d}{dx}(x^n) = nx^{n-1}$,
$\frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

When $a = 8$ and $x = 2$

$\frac{dy}{dx} = \frac{\sqrt{8}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}} = 2$

Using $y - y_1 = m(x - x_1)$, the tangent to C at P is

$y - 8 = 2(x - 2) = 2x - 4$

$y = 2x + 4$

The method for obtaining the equation of a straight line when you know its gradient and one point which it passes through is given in Chapter 5 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 1.

c At X, $y = 0 \Rightarrow 0 = 2x + 4 \Rightarrow x = -2$

So $OX = 2$

At Y, $x = 0 \Rightarrow y = 2 \times 0 + 4 \Rightarrow y = 4$

So $OY = 4$

Area $\triangle OXY = \frac{1}{2} OX \times OY = \frac{1}{2} \times 2 \times 4 = 4$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 51

Question:

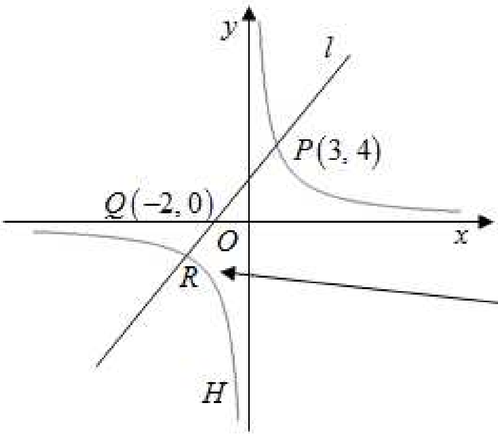
The point P with coordinates (3, 4) lies on the rectangular hyperbola H with equation $xy = 12$. The point Q has coordinates $(-2,\ 0)$. The points P and Q lie on the line l.

a Find an equation of l, giving your answer in the form $y = mx + c$, where m and c are real constants.

The line l cuts H at the point R, where P and R are distinct points.

b Find the coordinates of R.

Solution:



You can see from the diagram that both the x- and y-coordinates of R are negative. This will help you check your work in part (b).

a Using $\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$ with

$P(x_1, y_1)=(3, 4)$ and $Q(x_2, y_2)=(-2, 0)$.

$$\frac{y-4}{0-4}=\frac{x-3}{-2-3}$$

$$\frac{y-4}{-4}=\frac{x-3}{-5}$$

$$5(y-4)=4(x-3)$$

$$5y-20=4x-12$$

$$5y=4x+8$$

$$y=\frac{4}{5}x+\frac{8}{5} \quad \ldots\ldots \ast$$

Methods for finding the equation of a straight line are given in Chapter 5 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 1. You can use any correct method for finding the line.

b Substitute $\ast$ into the equation of H.

$$xy=12$$

$$x\left(\frac{4}{5}x+\frac{8}{5}\right)=12$$

$$\frac{4}{5}x^2+\frac{8}{5}x=12 \Rightarrow 4x^2+8x=60$$

$$x^2+2x-15=0$$

$$(x+5)(x-3)=0$$

$$x=-5, 3$$

$x=3$ corresponds to the point P.

For R, $x=-5$

$$y=\frac{12}{x}=\frac{12}{-5}=-2.4$$

The coordinates of R are $(-5, -2.4)$

Part (b) involves solving a pair of simultaneous equations, one of which is linear and one of which is quadratic, as xy is a quadratic term. Substitution is used to solve such pairs of equations.

There are two points on both the line l and the parabola H. You are given the coordinates of one $(3, 4)$ in the question. You are looking for the other point, so you must reject the value of x (3) which belongs to P and take the other value.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 52

Question:

The point $P(12, 3)$ lies on the rectangular hyperbola H with equation $xy = 36$.

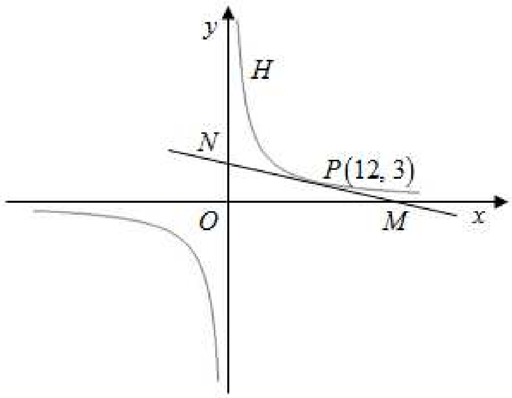
a Find an equation of the tangent to H at P.

The tangent to H at P cuts the x-axis at the point M and the y-axis at the point N.

b Find the length MN, giving your answer as a simplified surd.

Solution:

a



$$y = \frac{36}{x} = 36x^{-1}$$

$$\frac{dy}{dx} = -36x^{-2} = -\frac{36}{x^2}$$

At P, $x = 12$

$$\left(\frac{dy}{dx}\right)_{12} = -\frac{36}{12^2} = -\frac{1}{4}$$

Using $\frac{d}{dx}(x^n) = nx^{n-1}$,

$$\frac{d}{dx}(x^{-1}) = -1 \times x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

Using $y - y_1 = m(x - x_1)$, the tangent to H at P is

$$y - 3 = -\frac{1}{4}(x - 12)$$

$$4y - 12 = -x + 12$$

$$x + 4y = 24$$

No particular form of the equation of the tangent has been specified and any form would be accepted. $x + 4y = 24$ has been chosen here as, reading ahead, you will have to substitute $x = 0$ and $y = 0$ into the equation to find the corresponding y and x. It is very easy to do this with this equation. Reading ahead can often save time.

b At M, $y = 0 \Rightarrow x = 24 \Rightarrow OM = 24$

At M, $x = 0 \Rightarrow y = 6 \Rightarrow ON = 6$

$$MN^2 = OM^2 + ON^2$$

$$= 24^2 + 6^2 = 612 = 36 \times 7$$

$$MN = 6\sqrt{7}$$

A surd in its simplest form has the square root of the smallest possible single number.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 53

Question:

The point $P(5, 4)$ lies on the rectangular hyperbola H with equation $xy = 20$. The line l is the normal to H at P.

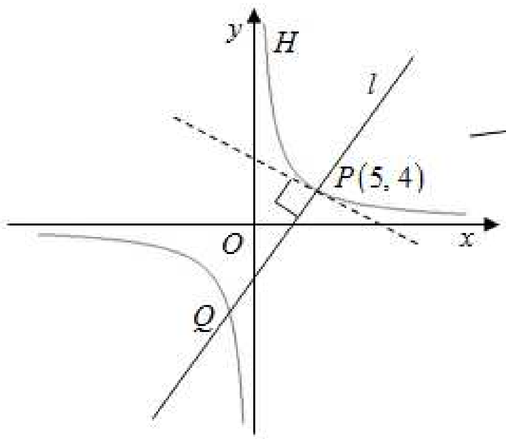
a Find an equation of l, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

The line l meets H again at the point Q.

b Find the coordinates of Q.

Solution:

a



The normal to H at P is perpendicular to the tangent at P. The tangent has here been drawn as a dotted line. To work out perpendicular gradients you will need the formula $mm' = -1$.

$$y = \frac{20}{x} = 20x^{-1}$$

$$\frac{dy}{dx} = -20x^{-2} = -\frac{20}{x^2}$$

At P, $x = 5$

$$\left(\frac{dy}{dx}\right)_5 = -\frac{20}{5^2} = -\frac{4}{5}$$

You have to find the gradient of the tangent before you can find the gradient of the normal. You find the gradient of the tangent by differentiating.

For the gradient of the normal, using $mm' = -1$,

$$\left(-\frac{4}{5}\right)m' = -1 \Rightarrow m' = \frac{5}{4}$$

Using $y - y_1 = m(x - x_1)$, the normal

to H at P is $y - 4 = \frac{5}{4}(x - 5)$

$$4(y-4) = 5(x-5)$$

$$4y - 16 = 5x - 25$$

$$5x - 4y - 9 = 0$$

You were asked to give your answer in the form $ax + by + c = 0$, where a, b and c are integers. The answer $-5x + 4y + 9 = 0$ would also have been acceptable.

b Rearranging the answer to part (a)

$$x = \frac{4y+9}{5}$$

Substitute this expression for x into $xy = 20$

$$\left(\frac{4y+9}{5}\right)y = 20$$

$$(4y+9)y = 100$$

$$4y^2 + 9y - 100 = 0$$

$$(y-4)(4y+25) = 0$$

$$y = 4,\ -\frac{25}{4}$$

Expressions like $4y^2 + 9y - 100$ are not easy to factorise but, as P lies on both l and H, you know that the y-coordinate of P, $y = 4$, must be one answer to the equation. So $(y-4)$ has to be one factor and the other can just be written down using $y \times 4y = 4y^2$ and $-4 \times +25 = -100$.

$y = 4$ corresponds to the point P.

For Q, $y = -6.25 \Rightarrow x(-6.25) = 20 \Rightarrow x = -\frac{20}{6.25} = -3.2$

The coordinates of Q are $(-3.2, -6.25)$.

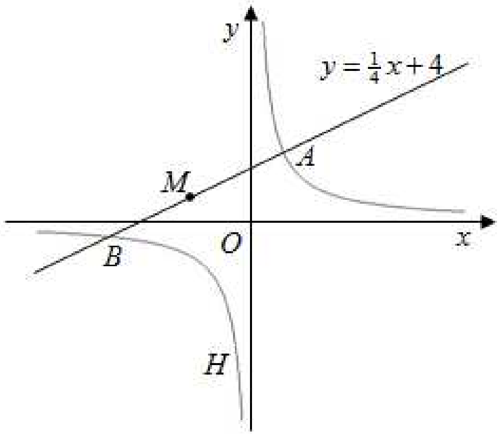
Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 54

Question:

The curve H with equation $x = 8t,\ y = \frac{8}{t}$ intersects the line with equation $y = \frac{1}{4}x + 4$ at the points A and B. The mid-point of AB is M. Find the coordinates of M.

Solution:



Substitute $x = 8t,\ y = \frac{8}{t}$ into $y = \frac{1}{4}x + 4$

$$\frac{8}{t} = \frac{1}{4} \times 8t + 4$$

$$\frac{8}{t} = 2t + 4$$

$\left(8t, \frac{8}{t}\right)$ is a general point on the rectangular hyperbola. The points A and B must be of this form and, if they also lie on the line with equation $y = \frac{1}{4}x + 4$, the points on the parabola must also satisfy the equation of the line.

Multiplying by t and rearranging

$$2t^2 + 4t - 8 = 0$$

$(\div 2)$ $\quad t^2 + 2t - 4 = (t+4)(t-2) = 0$

$$t = 2, -4$$

For A, say, $t = 2 \Rightarrow x = 8t = 8 \times 2 = 16$

and $y = \frac{8}{t} = \frac{8}{2} = 4$

The coordinates of A are $(16, 4)$

For B, say, $t = -4 \Rightarrow x = 8t = 8 \times -4 = -32$

and $y = \frac{8}{t} = \frac{8}{-4} = -2$

The coordinates of B are $(-32, -2)$

The question does not tell you which point is A and which point is B but, as the mid-point is not affected by the choice, it does not matter which is which and you can make your own choice.

The x-coordinate of the mid-point of AB is given by

$$x_M = \frac{16 - 32}{2} = -8$$

The y-coordinate of the mid-point of AB is given by

$$y_M = \frac{4 - 2}{2} = 1$$

The coordinates of M are $(-8, 1)$.

The coordinates of the mid-point M of $A(x_1, y_1)$ and $B(x_2, y_2)$ are given by $(x_M, y_M) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 55

Question:

The point $P(24t^2,\ 48t)$ lies on the parabola with equation $y^2 = 96x$. The point P also lies on the rectangular hyperbola with equation $xy = 144$.

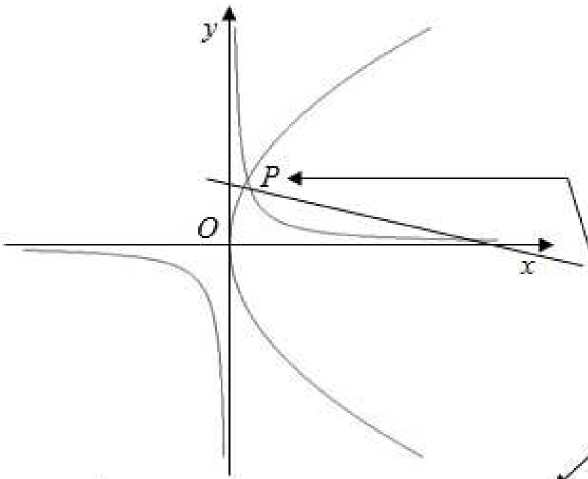
a Find the value of t and, hence, the coordinates of P.

b Find an equation of the tangent to the parabola at P, giving your answer in the form $y = mx + c$, where m and c are real constants.

c Find an equation of the tangent to the rectangular hyperbola at P, giving your answer in the form $y = mx + c$, where m and c are real constants.

Solution:

a



The point with coordinates $(at^2, 2at)$ always lies on the parabola with equation $y^2 = 4ax$, in this case $a = 24$, so P is on the parabola for all t. There will however only be one value of t for which P also lies on the rectangular hyperbola and you find it by substituting $(24t^2, 48t)$ into $xy = 144$.

$(24t^2, 48t)$ must satisfy the equation $xy = 144$

$$24t^2 \times 48t = 144$$

$$t^3 = \frac{144}{24 \times 48} = \frac{1}{8} \Rightarrow t = \frac{1}{2}$$

For P, $x = 24t^2 = 24 \times \left(\frac{1}{2}\right)^2 = 6$

$$y = 48t = 48 \times \frac{1}{2} = 24$$

The coordinates of P are $(6, 24)$.

b

$$y^2 = 96x \Rightarrow y = (96)^{\frac{1}{2}} x^{\frac{1}{2}} = 4\sqrt{6}x^{\frac{1}{2}}$$

$96 = 16 \times 6 = 4^2 \times 6$.
So $\sqrt{96} = \sqrt{(4^2 \times 6)} = 4\sqrt{6}$

$$\frac{dy}{dx} = \frac{1}{2} \times 4\sqrt{6}x^{-\frac{1}{2}} = \frac{2\sqrt{6}}{x^{\frac{1}{2}}}$$

At $x = 6$, $\frac{dy}{dx} = \frac{2\sqrt{6}}{\sqrt{6}} = 2$

Using $y - y_1 = m(x - x_1)$, an equation of the tangent to the parabola at P is

$$y - 24 = 2(x - 6) = 2x - 12$$

$$y = 2x + 12$$

c

$$y = \frac{144}{x} = 144x^{-1}$$

$$\frac{dy}{dx} = -144x^{-2} = -\frac{144}{x^2}$$

At $x = 6$, $\frac{dy}{dx} = -\frac{144}{6^2} = -4$

Using $y - y_1 = m(x - x_1)$, an equation of the tangent to the hyperbola at P is

$$y - 24 = -4(x - 6) = -4x + 24$$

$$y = -4x + 48$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 56

Question:

The points $P(9, 8)$ and $Q(6, 12)$ lie on the rectangular hyperbola H with equation $xy = 72$.

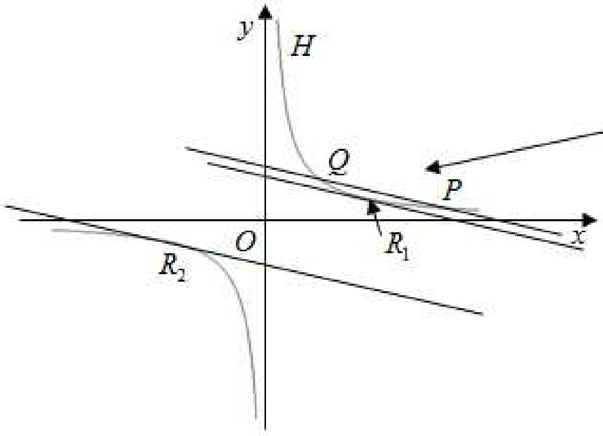
a Show that an equation of the chord PQ of H is $4x + 3y = 60$.

The point R lies on H. The tangent to H at R is parallel to the chord PQ.

b Find the exact coordinates of the two possible positions of R.

Solution:

a



The diagram shows that there are two positions of R, labelled R_1 and R_2 in the diagram, where the tangents to H are parallel to PQ.

Using $\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$ with

$P(x_1, y_1)=(9, 8)$ and $Q(x_2, y_2)=(6, 12)$,

an equation of the chord PQ is

$$\frac{y-8}{12-8}=\frac{x-9}{6-9}$$

$$\frac{y-8}{4}=\frac{x-9}{-3}$$

$$-3(y-8)=4(x-9)$$

$$-3y+24=4x-36$$

$4x+3y=60$, as required.

When you are asked to show that an equation is true, you must use algebra to transform your equation to the equation exactly as it is printed in the question.

b Rearranging the answer to part (a)

$$3y=-4x+60 \Rightarrow y=-\frac{4}{3}x+20$$

The gradient of the chord is $-\frac{4}{3}$.

If the tangents are parallel to AB, the gradients of the tangents must also be $-\frac{4}{3}$.

The lines $y=mx+c$ and $y=m'x+c'$ are parallel if $m=m'$. For AB, $m=-\frac{4}{3}$. For a tangent, $m'=\frac{dy}{dx}$. The key step is, therefore, solving $\frac{dy}{dx}=-\frac{4}{3}$.

$$y=72x^{-1} \Rightarrow \frac{dy}{dx}=-72x^{-2}=-\frac{72}{x^2}$$

$$-\frac{72}{x^2}=-\frac{4}{3}$$

$$72\times 3=4x^2 \Rightarrow x^2=\frac{72\times 3}{4}=54$$

$$x=\pm\sqrt{54}=\pm 3\sqrt{6}$$

At R_1, $x=3\sqrt{6}$, $y=\frac{72}{3\sqrt{6}}=\frac{12\times 6}{3\sqrt{6}}=4\sqrt{6}$

At R_2, $x=-3\sqrt{6}$, $y=-\frac{72}{3\sqrt{6}}=-\frac{12\times 6}{3\sqrt{6}}=-4\sqrt{6}$

The coordinates of the two possible positions of R are $(3\sqrt{6}, 4\sqrt{6})$ and $(-3\sqrt{6}, -4\sqrt{6})$.

Solutionbank FP1
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Review Exercise
Exercise A, Question 57

Question:

A rectangular hyperbola H has cartesian equation $xy = 9$. The point $\left(3t, \frac{3}{t}\right)$ is a general point on H.

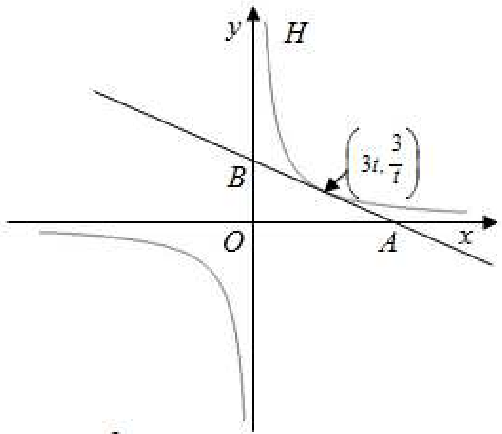
a Show that an equation of the tangent to H at $\left(3t, \frac{3}{t}\right)$ is $x + t^2 y = 6t$.

The tangent to H at $\left(3t, \frac{3}{t}\right)$ cuts the x-axis at A and the y-axis at B. The point O is the origin of the coordinate system.

b Show that, as t varies, the area of the triangle OAB is constant.

Solution:

a



$y = \frac{9}{x} = 9x^{-1}$

$\frac{dy}{dx} = -9x^{-2} = -\frac{9}{x^2}$

Using $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$,
$\frac{d}{dx}\left(9x^{-1}\right) = -1 \times 9x^{-1-1} = -9x^{-2} = -\frac{9}{x^2}$.

When $x = 3t$, $\frac{dy}{dx} = -\frac{9}{(3t)^2} = -\frac{1}{t^2}$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(3t, \frac{3}{t}\right)$,

the tangent to H

$$y - \frac{3}{t} = -\frac{1}{t^2}(x - 3t)$$

$(\times t^2)$ $\quad t^2 y - 3t = -x + 3t$

$x + t^2 y = 6t$, as required.

You can use $(x_1, y_1) = \left(3t, \frac{3}{t}\right)$ in the formula $y - y_1 = m(x - x_1)$ in exactly the same way as you use coordinates with numerical values like, say, $(6, 4)$.

b For A, substitute, $y = 0$ into $x + t^2 y = 6t$.

$x = 6t \Rightarrow OA = 6t$

For B, substitute, $x = 0$ into $x + t^2 y = 6t$.

$t^2 y = 6t \Rightarrow y = \frac{6}{t} \Rightarrow OB = \frac{6}{t}$

Area $\triangle OAB = \frac{1}{2} OA \times OB = \frac{1}{2} \times 6t \times \frac{6}{t} = 18$

This area, 18, is a constant independent of t.

This result means that no matter which point you take on this rectangular hyperbola the area of the triangle OAB is always the same, 18.

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Review Exercise
Exercise A, Question 58

Question:

The point $P\left(ct, \frac{c}{t}\right)$ lies on the hyperbola with equation $xy = c^2$, where c is a positive constant.

a Show that an equation of the normal to the hyperbola at P is

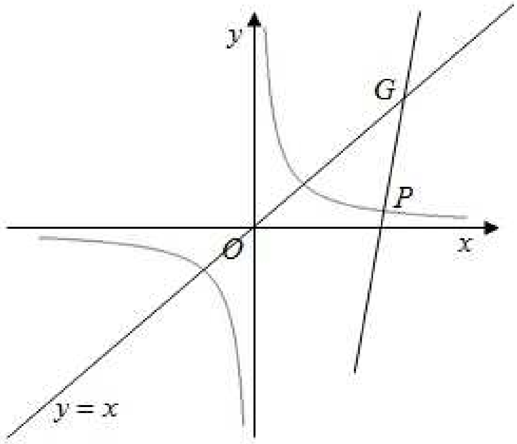
$t^3x - ty - c(t^4 - 1) = 0$.

The normal to the hyperbola at P meets the line $y = x$ at G. Given that $t \neq \pm 1$,

b show that $PG^2 = c^2\left(t^2 + \frac{1}{t^2}\right)$.

Solution:

a



$$y = \frac{c^2}{x} = c^2x^{-1}$$

$$\frac{dy}{dx} = -c^2x^{-2} = -\frac{c^2}{x^2}$$

At P, $x = ct$

$$\frac{dy}{dx} = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$$

For the gradient of the normal, using $mm' = -1$,

$$\left(-\frac{1}{t^2}\right)m' = -1 \Rightarrow m' = t^2$$

The normal to H at P is perpendicular to the tangent at P. To work out perpendicular gradients you will need the formula $mm' = -1$. So you have to find the gradient of the tangent before you can find the gradient of the normal. You find the gradient of the tangent using differentiation.

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(ct, \frac{c}{t}\right)$,

an equation of the normal to the hyperbola at P is

$$y - \frac{c}{t} = t^2(x - ct)$$

$$y - \frac{c}{t} = t^2x - ct^3$$

$$(\times t) \quad yt - c = t^3x - ct^4$$

$$t^3x - ty - ct^4 + c = 0$$

$t^3x - ty - c(t^4 - 1) = 0$, as required

When you are asked to show that an equation is true, you must use algebra to transform your equation to the equation exactly as it is printed in the question.

b For G, substitute $y = x$ into the result in part (a)

$$t^3x - tx - c(t^4 - 1) = 0$$

$$(t^3 - t)x = c(t^4 - 1)$$

$$x = \frac{c(t^4-1)}{t^3-t} = \frac{c\cancel{(t^2-1)}(t^2+1)}{t\cancel{(t^2-1)}} = \frac{ct^2+c}{t} = ct + \frac{c}{t}$$

You could not "cancel" the (t^2-1) terms if $t = \pm 1$, as then (t^2-1) would be 0, but these cases are explicitly ruled out in the question.

The coordinates of G are $\left(ct+\frac{c}{t}, ct+\frac{c}{t}\right)$

$$PG^2 = (x_1-x_2)^2 + (y_1-y_2)^2$$

$$= \left(ct+\frac{c}{t}-ct\right)^2 + \left(ct+\frac{c}{t}-\frac{c}{t}\right)^2$$

$$= \frac{c^2}{t^2} + c^2t^2 = c^2\left(t^2+\frac{1}{t^2}\right), \text{ as required.}$$

Using $d^2 = (x_1-x_2)^2 + (y_1-y_2)^2$ with $(x_1, y_1) = \left(ct+\frac{c}{t}, ct+\frac{c}{t}\right)$ and $(x_2, y_2) = \left(ct, \frac{c}{t}\right)$

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Review Exercise
Exercise A, Question 59

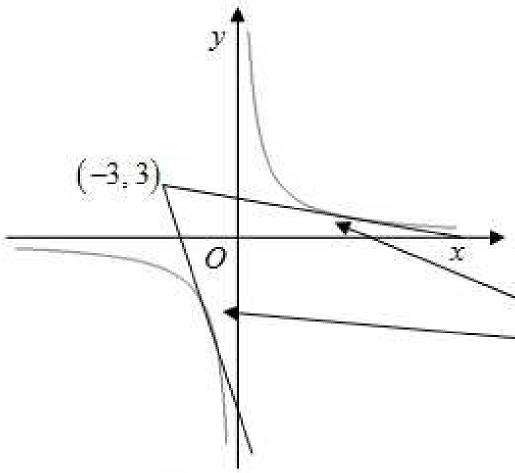
Question:

a Show that an equation of the tangent to the rectangular hyperbola with equation $xy = c^2$ at the point $\left(ct,\ \frac{c}{t}\right)$ is $t^2y + x = 2ct$.

Tangents are drawn from the point $(-3,\ 3)$ to the rectangular hyperbola with equation $xy = 16$.

b Find the coordinates of the points of contact of these tangents with the hyperbola.

Solution:



The diagram shows that, in part (b), the tangents have two points of contact with the hyperbola. One is in the first quadrant and the other in the third.

a $y = \frac{c^2}{x} = c^2 x^{-1}$

$$\frac{dy}{dx} = -c^2 x^{-2} = -\frac{c^2}{x^2}$$

At $x = ct$

$$\frac{dy}{dx} = -\frac{c^2}{c^2 t^2} = -\frac{1}{t^2}$$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(ct, \frac{c}{t}\right)$,

an equation of the tangent to the hyperbola is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$y - \frac{c}{t} = -\frac{x}{t^2} + \frac{c}{t}$$

$$y + \frac{x}{t^2} = \frac{2c}{t}$$

$(\times t^2)$ $t^2 y + x = 2ct$, as required.

Part (a) is a general question. Part (b) is about the specific rectangular hyperbola with $c^2 = 16$. The first step in part (b) is to adapt the answer in (a) to (b) by substituting $c = 4$.

b When $c = 4$, the equation of the tangent is

$$t^2 y + x = 8t$$

$(-3, 3)$ satisfies the equation

$$3t^2 - 3 = 8t$$

$(-3, 3)$ must lie on both tangents and you use this to obtain a quadratic in t.

$$3t^2 - 8t - 3 = (3t+1)(t-3) = 0$$
$$t = -\frac{1}{3}, 3$$

The points on the hyperbola are $\left(4t, \frac{4}{t}\right)$

When $t = -\frac{1}{3}$, the point is $\left(-\frac{4}{3}, \frac{4}{-\frac{1}{3}}\right) = \left(-\frac{4}{3}, -12\right)$

When $t = 3$, the point is $\left(12, \frac{4}{3}\right)$

The points of contact of the tangents with the hyperbola are $\left(-\frac{4}{3}, -12\right)$ and $\left(12, \frac{4}{3}\right)$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

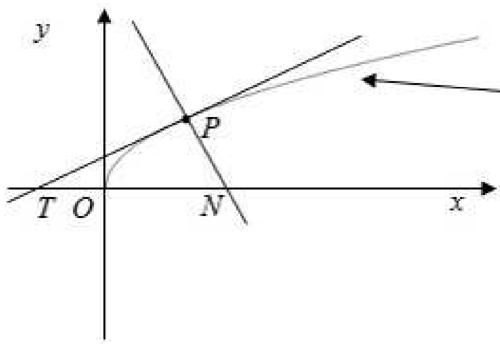
Review Exercise
Exercise A, Question 60

Question:

The point $P(at^2,\ 2at)$, where $t > 0$, lies on the parabola with equation $y^2 = 4ax$.

The tangent and normal at P cut the x-axis at the points T and N respectively. Prove that $\frac{PT}{PN} = t$.

Solution:



The $t > 0$ in the question implies that you only need to consider the part of the parabola where $y > 0$, that is the part above the x-axis.

To find an equation of the tangent PT

$y^2 = 4ax \Rightarrow y = 2a^{\frac{1}{2}}x^{\frac{1}{2}} \quad (y > 0)$

$$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} \times 2a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

Using $\frac{d}{dx}(x^n) = nx^{n-1}$, $\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

At $x = at^2$, $x^{\frac{1}{2}} = a^{\frac{1}{2}}t$

$$\frac{dy}{dx} = \frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}t} = \frac{1}{t}$$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = (at^2, 2at)$, an equation of the tangent to the parabola at P is

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$ty - 2at = x - at^2$$

$$ty = x + at^2 \quad \ldots \ldots \; ❶$$

To find the x-coordinate of T, substitute $y = 0$ into ❶

The tangent crosses the x-axis where $y = 0$.

$$0 = x + at^2 \Rightarrow x = -at^2$$

Using $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ with $(x_1, y_1) = (at^2, 2at)$ and $(x_2, y_2) = (-at^2, 0)$

$$PT^2 = \left(at^2 - (-at^2)\right)^2 + (2at - 0)^2$$

$$= (2at^2)^2 + 4a^2t^2 = 4a^2t^4 + 4a^2t^2$$

$$= 4a^2t^2(t^2 + 1) \quad \ldots \ldots \; ❷$$

To find an equation of the normal PN.

Using $mm' = -1$,

$$\frac{1}{t} \times m' = -1 \Rightarrow m' = -t$$

The normal is perpendicular to the tangent. From working earlier in the question, you know that the gradient of the tangent is $\frac{1}{t}$.

Using $y - y_1 = m'(x - x_1)$ with $(x_1, y_1) = (at^2, 2at)$, an equation of the normal to the parabola at P is

$$y - 2at = -t\left(x - at^2\right)$$
$$= -tx + at^3$$
$$y + tx = 2at + at^3 \quad \ldots \ldots \text{❸}$$

The normal crosses the x-axis where $y = 0$.

To find the x-coordinate of N, substitute $y = 0$ into ❸

$$tx = 2at + at^3 \Rightarrow x = 2a + at^2$$

Using $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ with

$(x_1, y_1) = \left(at^2, 2at\right)$ and $(x_2, y_2) = \left(2a + at^2, 0\right)$

$$PN^2 = \left(at^2 - \left(2a + at^2\right)\right)^2 + (2at - 0)^2$$
$$= (2a)^2 + (2at)^2 = 4a^2 + 4a^2t^2$$
$$= 4a^2\left(1 + t^2\right) \quad \ldots \ldots \text{❹}$$

From ❷ and ❹

$$\frac{PT^2}{PN^2} = \frac{4a^2t^2\left(t^2 + 1\right)}{4a^2\left(t^2 + 1\right)} = t^2$$

Hence

$\frac{PT}{PN} = t$, as required.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 61

Question:

The point P lies on the parabola with equation $y^2 = 4ax$, where a is a positive constant.

a Show that an equation of the tangent to the parabola $P(ap^2,\ 2ap),\ p > 0$, is $py = x + ap^2$.

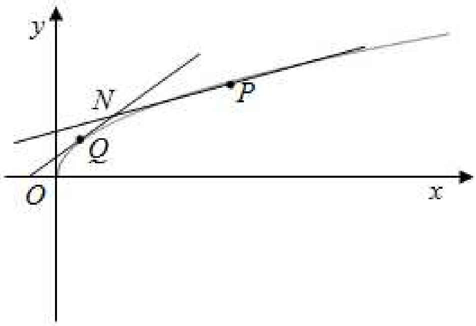
The tangents at the points $P(ap^2,\ 2ap)$ and $Q(aq^2,\ 2aq)(p \neq q,\ p > 0,\ q > 0)$ meet at the point N.

b Find the coordinates of N.

Given further that N lies on the line with equation $y = 4a$,

c find p in terms of q.

Solution:



a $y^2 = 4ax \Rightarrow y = 2a^{\frac{1}{2}}x^{\frac{1}{2}} \quad (y > 0)$

$$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} \times 2a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

At $x = ap^2$, $x^{\frac{1}{2}} = a^{\frac{1}{2}}p$

$$\frac{dy}{dx} = \frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}p} = \frac{1}{p}$$

Using $\frac{d}{dx}(x^n) = nx^{n-1}$,
$\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = (ap^2, 2ap)$,
an equation of the tangent to the parabola at P is

$$y - 2ap = \frac{1}{p}(x - ap^2)$$

$$py - 2ap^2 = x - ap^2$$

$$py = x + ap^2, \text{ as required.}$$

b An equation of the tangent to the parabola at Q is

$$qy = x + aq^2 \quad \dots \dots \text{ ➊}$$

$$py = x + ap^2 \quad \dots \dots \text{ ➋}$$

The equation of the tangent at Q is the same as the equation of the tangent at P with the ps replaced by qs. You do not have to work out the equation again.

➋ − ➊ $\quad py - qy = ap^2 - aq^2$

$$(p - q)y = a(p^2 - q^2)$$

$$y = \frac{a(p^2 - q^2)}{p - q} = \frac{a\cancel{(p-q)}(p+q)}{\cancel{p-q}} = a(p+q)$$

To find x and y, in terms of p and q, you solve equations ➊ and ➋ as a pair of simultaneous linear equations.

Substitute into ➊

$$qa(p+q) = x + aq^2$$

$$x = qa(p+q) - aq^2 = apq + aq^2 - aq^2 = apq$$

The coordinates of N are $(apq, a(p+q))$.

c If N lies on $y = 4a$,

$$\cancel{a}(p+q) = 4\cancel{a}$$

$$p = 4 - q$$

If N lies on the line with equation $y = 4a$, then the y-coordinate of N, which is $a(p+q)$, must be $4a$.

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Review Exercise
Exercise A, Question 62

Question:

The point $P(at^2,\ 2at),\ t \neq 0$ lies on the parabola with equation $y^2 = 4ax$, where a is a positive constant.

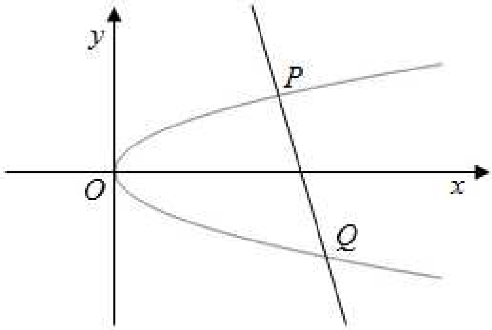
a Show that an equation of the normal to the parabola at P is

$y + xt = 2at + at^3$.

The normal to the parabola at P meets the parabola again at Q.

b Find, in terms of t, the coordinates of Q.

Solution:



a $y^2 = 4ax \Rightarrow y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$

$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2} \times 2a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$

At $x = at^2$, $x^{\frac{1}{2}} = a^{\frac{1}{2}}t$

$\frac{dy}{dx} = \frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}t} = \frac{1}{t}$

Using $mm' = -1$,

$\frac{1}{t} \times m' = -1 \Rightarrow m' = -t$

The normal is perpendicular to the tangent, so you must first find the gradient of the tangent. Then you use $mm' = -1$ to find the gradient of the normal.

Using $y - y_1 = m'(x - x_1)$ with $(x_1, y_1) = (at^2, 2at)$,

an equation of the normal to the parabola at P is

$y - 2at = -t(x - at^2)$

$= -tx + at^3$

$y + tx = 2at + at^3$, as required.

When you are asked to show that an equation is true, you must use algebra to transform your equation to the equation exactly as it is printed in the question.

b Let the coordinates of Q be $(aq^2, 2aq)$

The point Q lies on the normal at P, so

You substitute $x = aq^2$ and $y = 2aq$ into the answer to part (a).

$2aq + taq^2 = 2at + at^3$

$2aq - 2at + atq^2 - at^3 = 0$

$2a(q - t) + at(q^2 - t^2) = 0$

$2a(q - t) + at(q - t)(q + t) = 0$

$a(q - t)(2 + t(q + t)) = 0$

$2 + tq + t^2 = 0$

$q = -\frac{t^2 + 2}{t}$

There are two possibilities here: $q - t = 0$ and $2 + t(q + t) = 0$. As P and Q are different points, $q \neq t$, so you need only consider the second possibility. You use this to find q in terms of t.

The coordinates of Q are $\left(a\left(\frac{t^2+2}{t}\right)^2, -2a\left(\frac{t^2+2}{t}\right)\right)$

Replace the q in $(aq^2, 2aq)$ by $-\frac{t^2+2}{t}$. You need not attempt to simplify this further.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 63

Question:

a Show that the normal to the rectangular hyperbola $xy = c^2$, at the point $P\left(ct,\ \frac{c}{t}\right)$, $t \neq 0$, has equation $y = t^2x + \frac{c}{t} - ct^3$.

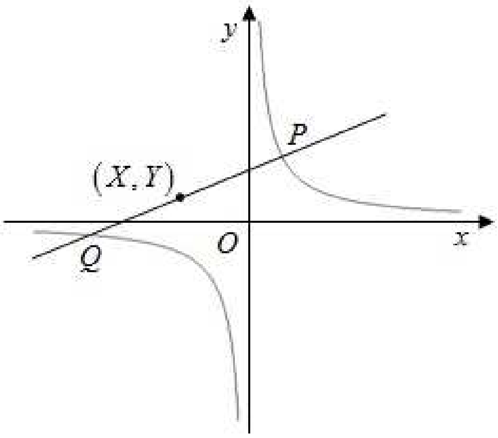
The normal to the hyperbola at P meets the hyperbola again at the point Q.

b Find, in terms of t, the coordinates of the point Q.

Given that the mid-point of PQ is (X, Y) and that $t \neq \pm 1$,

c show that $\frac{X}{Y} = -\frac{1}{t^2}$.

Solution:



a $y = \dfrac{c^2}{x} = c^2x^{-1}$

$\dfrac{dy}{dx} = -c^2x^{-2} = -\dfrac{c^2}{x^2}$

At P, $x = ct$

$\dfrac{dy}{dx} = -\dfrac{c^2}{c^2t^2} = -\dfrac{1}{t^2}$

For the gradient of the normal, using $mm' = -1$,

$\left(-\dfrac{1}{t^2}\right)m' = -1 \Rightarrow m' = t^2$

The normal to H at P is perpendicular to the tangent at P. To work out perpendicular gradients you will need the formula $mm' = -1$. So you have to find the gradient of the tangent before you can find the gradient of the normal. You find the gradient of the tangent by differentiating.

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(ct, \dfrac{c}{t}\right)$,

an equation of the normal to the hyperbola at P is

$y - \dfrac{c}{t} = t^2(x - ct)$

$= t^2x - ct^3$

$y = t^2x + \dfrac{c}{t} - ct^3$, as required. … ❶

When you are asked to show that an equation is true, you must use algebra to transform your equation to the equation exactly as it is printed in the question. In this case, the form of the printed equation suggests a method for the next part of the question.

b $xy = c^2 \Rightarrow y = \dfrac{c^2}{x}$ … … ❷

For Q, from ❶ and ❷

$t^2x + \dfrac{c}{t} - ct^3 = \dfrac{c^2}{x}$

$\times t$ and collect terms as a quadratic in x

$t^3x^2 + (c - ct^4)x - c^2t = 0$

$(x - ct)(t^3x + c) = 0$

$x = ct$ corresponds to P

Writing the equation of the rectangular hyperbola, in the form $y = \ldots$, enables you to eliminate y quickly between ❶ and ❷.

The x-coordinate of P, ct, must be a solution of the equation. So $(x - ct)$ must be a factor of the quadratic and, so, you can write down the second factor using $x \times t^3x = t^3x^2$ and $-ct \times c = -c^2t$.

For Q, $x = -\frac{c}{t^3}$

Substitute the x-coordinate into ❷

$$y = \frac{c^2}{x} = \frac{c^2}{-\frac{c}{t^3}} = -ct^3$$

The coordinates of Q are $\left(-\frac{c}{t^3}, -ct^3\right)$

c

$$X = \frac{ct + \left(-\frac{c}{t^3}\right)}{2} = \frac{ct^4 - c}{2t^3} = \frac{c\left(t^4 - 1\right)}{2t^3}$$

$$Y = \frac{\frac{c}{t} + \left(-ct^3\right)}{2} = \frac{ct - ct^4}{2t} = \frac{c\left(1 - t^4\right)}{2t}$$

$$\frac{X}{Y} = \frac{\frac{c\left(t^4-1\right)}{2t^3}}{\frac{c\left(1-t^4\right)}{2t}} = \frac{c\left(t^4-1\right)}{2t^3} \times \frac{2t}{c\left(1-t^4\right)}$$

$= -\frac{1}{t^2}$, as required.

The coordinates of the mid-point of $A(x_1, y_1)$ and $B(x_2, y_2)$ are given by $(X, Y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Multiplying all terms on the top and bottom of the fraction by t^3.

Multiplying all terms on the top and bottom of the fraction by t.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 64

Question:

The rectangular hyperbola C has equation $xy = c^2$, where c is a positive constant.

a Show that the tangent to C at the point $P\left(cp,\ \frac{c}{p}\right)$ has equation $p^2y = -x + 2cp$.

The point Q has coordinates $Q\left(cq,\ \frac{c}{q}\right),\ q \neq p$.

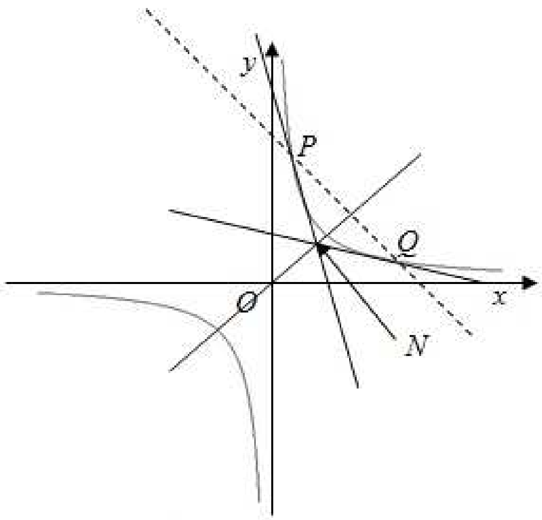
The tangents to C at P and Q meet at N. Given that $p + q \neq 0$,

b show that the y-coordinate of N is $\frac{2c}{p+q}$.

The line joining N to the origin O is perpendicular to the chord PQ.

c Find the numerical value of p^2q^2.

Solution:



a $y = \frac{c^2}{x} = c^2 x^{-1}$

$\frac{dy}{dx} = -c^2 x^{-2} = -\frac{c^2}{x^2}$

At $x = cp$

$\frac{dy}{dx} = -\frac{c^2}{c^2 p^2} = -\frac{1}{p^2}$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(cp, \frac{c}{p}\right)$,

an equation of the tangent to the hyperbola is

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$y - \frac{c}{p} = -\frac{x}{p^2} + \frac{c}{p}$

$y = -\frac{x}{p^2} + \frac{2c}{p}$

$(\times p^2)$ $p^2 y = -x + 2cp$, as required. … ❶

The equation of the tangent at Q is the same as the equation of the tangent at P with the ps replaced by qs. You do not have to work out the equation twice.

b The tangent at Q is

$q^2 y = -x + 2cq$ … … ❷

To find the y-coordinate of N subtract ❷ from ❶

$(p^2 - q^2)y = 2c(p - q)$

$y = \frac{2c(p-q)}{p^2 - q^2} = \frac{2c\cancel{(p-q)}}{\cancel{(p-q)}(p+q)} = \frac{2c}{p+q}$, as required.

To find y, you eliminate x from equations ❶ and ❷. These equations are a pair of simultaneous linear equations and the method of solving them is essentially the same as you learnt for GCSE.

c To find the x-coordinate of N substitute the result of part (b) into ➊

$$\frac{2cp^2}{p+q} = -x + 2cp$$

$$x = 2cp - \frac{2cp^2}{p+q} = \frac{2cp(p+q) - 2cp^2}{p+q} = \frac{2cpq}{p+q}$$

The gradient of PQ, m say, is given by

$$m = \frac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \frac{\frac{\cancel{c}\,\cancel{(q-p)}^{-1}}{pq}}{\cancel{c}\,\cancel{(p-q)}} = -\frac{1}{pq}$$

The gradient m is found using $m = \frac{y_2 - y_1}{x_2 - x_1}$ with $(x_1, y_1) = \left(cp, \frac{c}{p}\right)$ and $(x_2, y_2) = \left(cq, \frac{c}{q}\right)$

The gradient of ON, m' say, is given by

$$m' = \frac{\frac{2c}{p+q}}{\frac{2cpq}{p+q}} = \frac{1}{pq}$$

Given that ON is perpendicular to PQ

$$mm' = -1$$

$$-\frac{1}{pq} \times \frac{1}{pq} = -1 \Rightarrow p^2q^2 = 1$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 65

Question:

The point P lies on the rectangular hyperbola $xy = c^2$, where c is a positive constant.

a Show that an equation of the tangent to the hyperbola at the point $P\left(cp, \frac{c}{p}\right)$, $p > 0$, is $yp^2 + x = 2cp$.

This tangent at P cuts the x-axis at the point S.

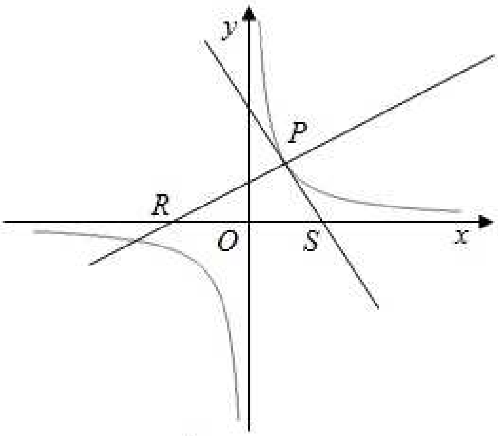
b Write down the coordinates of S.

c Find an expression, in terms of p, for the length of PS.

The normal at P cuts the x-axis at the point R. Given that the area of ΔRPS is $41c^2$,

d find, in terms of c, the coordinates of the point P.

Solution:



a $y = \frac{c^2}{x} = c^2 x^{-1}$

$\frac{dy}{dx} = -c^2 x^{-2} = -\frac{c^2}{x^2}$

At $x = cp$

$\frac{dy}{dx} = -\frac{c^2}{c^2 p^2} = -\frac{1}{p^2}$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(cp, \frac{c}{p}\right)$,

an equation of the tangent to the hyperbola is

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$y - \frac{c}{p} = -\frac{x}{p^2} + \frac{c}{p}$

$y + \frac{x}{p^2} = \frac{2c}{p}$

$(\times p^2)$ $p^2 y + x = 2cp$, as required. ... ❶

The tangent crosses the x-axis at $y = 0$. You can put $y = 0$ into ❶ in your head and just write down the coordinates of S. No working is needed.

b $(2cp, 0)$

c Using $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ with

$(x_1, y_1) = \left(cp, \frac{c}{p}\right)$ and $(x_2, y_2) = (2cp, 0)$

$PS^2 = (cp - 2cp)^2 + \left(\frac{c}{p} - 0\right)^2 = c^2 p^2 + \frac{c^2}{p^2}$

$= c^2\left(p^2 + \frac{1}{p^2}\right) = c^2\left(\frac{p^4 + 1}{p^2}\right)$

$PS = \frac{c}{p}(1 + p^4)^{\frac{1}{2}}$

There are many possible forms for this answer. Any equivalent form would gain full marks.

d To find the equation of the normal at P.

The working in part (a) shows the gradient of the tangent is $-\frac{1}{p^2}$.

Let the gradient of the normal be m'.

Using $mm' = -1$,

$$-\frac{1}{p^2} \times m' = -1 \Rightarrow m' = p^2$$

Using $y - y_1 = m'(x - x_1)$ with $(x_1, y_1) = \left(cp, \frac{c}{p}\right)$,

an equation of the normal to the hyperbola at P is

$$y - \frac{c}{p} = p^2(x - cp)$$

$$= p^2x - cp^3$$

$$p^2x = y - \frac{c}{p} + cp^3$$

To find an expression for the area of the triangle you can obtain the length of the side RS and use that as the base of the triangle in the formula for the area of the triangle. First you need to obtain an equation of the normal and use it to find the coordinates of R.

To find the x-coordinate of R, substitute $y = 0$

$$p^2x = -\frac{c}{p} + cp^3 \Rightarrow x = cp - \frac{c}{p^3}$$

$$RS = 2cp - \left(cp - \frac{c}{p^3}\right) = cp + \frac{c}{p^3} = c\left(\frac{p^4 + 1}{p^3}\right)$$

$$\text{Area } \triangle RPS = \frac{1}{2} RS \times \text{height}$$

If RS is taken as the base of the triangle, the height of the triangle is the y-coordinate of P.

$$41c^2 = \frac{1}{2} \times c\left(\frac{p^4 + 1}{p^3}\right) \times \frac{c}{p}$$

$$= \frac{c^2}{2p^4}(p^4 + 1)$$

$$82p^4 = p^4 + 1 \Rightarrow p^4 = \frac{1}{81} \Rightarrow p = \frac{1}{3}$$

As $p > 0$ is given in the question, you need not consider the alternative solution $p = -\frac{1}{3}$.

The coordinates of P are $\left(cp, \frac{c}{p}\right) = \left(\frac{c}{3}, 3c\right)$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 66

Question:

The curve C has equation $y^2 = 4ax$, where a is a positive constant.

a Show that an equation of the normal to C at the point $P(ap^2,\ 2ap),\ (p \neq 0)$ is $y + px = 2ap + ap^3$.

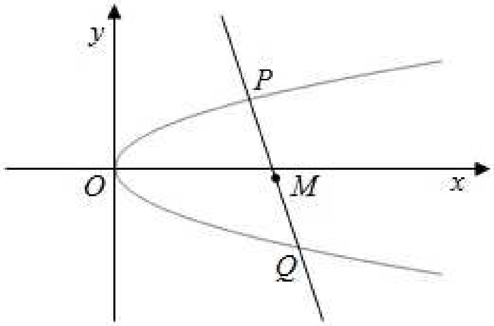
The normal at P meets C again at the point $Q(aq^2,\ 2aq)$.

b Find q in terms of p.

Given that the mid-point of PQ has coordinates $\left(\frac{125}{18}a\ ,\ -3a\right)$,

c use your answer to **b**, or otherwise, to find the value of p.

Solution:



a $y^2 = 4ax \Rightarrow y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$

$$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} \times 2a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

At $x = at^2$, $x^{\frac{1}{2}} = a^{\frac{1}{2}}p$

$$\frac{dy}{dx} = \frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}p} = \frac{1}{p}$$

Using $mm' = -1$,

$$\frac{1}{p} \times m' = -1 \Rightarrow m' = -p$$

The normal is perpendicular to the tangent, so you must first find the gradient of the tangent. Then you use $mm' = -1$ to find the gradient of the normal.

Using $y - y_1 = m'(x - x_1)$ with $(x_1, y_1) = (ap^2, 2ap)$,

an equation of the normal to the parabola at P is

$$y - 2ap = -p(x - ap^2)$$
$$= -px + ap^3$$
$$y + xp = 2ap + ap^3, \text{ as required.}$$

When you are asked to show that an equation is true, you must use algebra to transform your equation to the equation exactly as it is printed in the question.

b Let the coordinates of Q be $(aq^2, 2aq)$

The point Q lies on the normal at P, so

$$2aq + paq^2 = 2ap + ap^3$$
$$2aq - 2ap + apq^2 - ap^3 = 0$$
$$2a(q-p) + ap(q^2 - p^2) = 0$$
$$2a\cancel{(q-p)} + ap\cancel{(q-p)}(q+p) = 0$$
$$2 + p(q+p) = 0$$
$$2 + pq + p^2 = 0$$
$$pq = -p^2 - 2 \Rightarrow q = -p - \frac{2}{p}$$

As P and Q are different points, $p \neq q$ and it follows that $q - p \neq 0$. You can, therefore, divide throughout this line by $(q - p)$.

Any equivalent of this expression is acceptable, e.g. $q = -\frac{p^2+2}{p}$.

c The y-coordinate of the mid-point of PQ is given by

$$\frac{y_P + y_Q}{2} = \frac{2ap + 2aq}{2} = \frac{\cancel{2}a(p+q)}{\cancel{2}} = a(p+q)$$

You only need one equation to find p and so you do not need to consider both coordinates of the mid-point. Either would do, but it is sensible to choose the coordinate with the easier numbers. In this case, that is the y-coordinate.

The answer to part (b) is

$$q = -p - \frac{2}{p}$$

Therefore $\quad p + q = -\frac{2}{p}$

The y-coordinate of the mid-point is

$$a(p+q) = a \times -\frac{2}{p} = -3a \text{, given.}$$

$$-\frac{2a}{p} = -3a \Rightarrow p = \frac{2}{3}$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 67

Question:

The parabola C has equation $y^2 = 32x$.

a Write down the coordinates of the focus S of C.

b Write down the equation of the directrix of C.

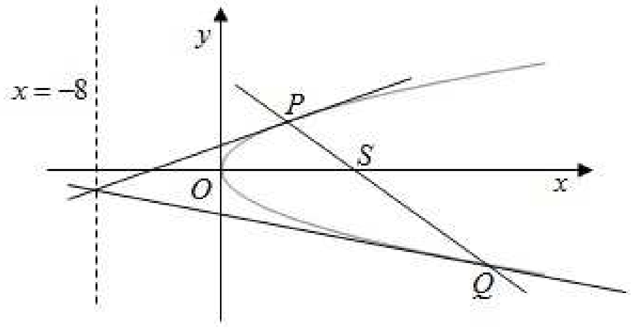
The points P (2, 8) and Q (32, −32) lie on C.

c Show that the line joining P and Q goes through S.

The tangent to C at P and the tangent to C at Q intersect at the point D.

d Show that D lies on the directrix of C.

Solution:



a $(8, 0)$

b $x = -8$

If $y^2 = 4ax$, the focus has coordinates $(a, 0)$ and the directrix has equation $x = -a$. Comparison of $y^2 = 4ax$ with $y^2 = 32x$, shows that, in this case, $a = 8$.

c Using $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$ with $(x_1, y_1) = (2, 8)$ and $(x_2, y_2) = (32, -32)$, an equation of PQ is

$$\frac{y-8}{-32-8} = \frac{x-2}{32-2}$$

$$\frac{y-8}{-4\cancel{0}} = \frac{x-2}{3\cancel{0}}$$

$$3y - 24 = -4x + 8$$

$$3y + 4x = 32$$

Substitute $y = 0$

$$0 + 4x = 32 \Rightarrow x = 8$$

The coordinates of S $(8, 0)$ satisfy the equation of PQ.

Hence S lies on the line joining P and Q.

Methods for finding the equation of a straight line are given in Chapter 5 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 1. You can use any correct method for finding the line.

d $y^2 = 32x \Rightarrow y = \pm 4\sqrt{2}x^{\frac{1}{2}}$

$\sqrt{32} = \sqrt{(16\times 2)} = \sqrt{16}\times\sqrt{2} = 4\sqrt{2}$

P is on the upper half of the parabola where $y = +4\sqrt{2}x^{\frac{1}{2}}$

On the upper half of the parabola, in the first quadrant, the y-coordinates of P are positive.

$$\frac{dy}{dx} = \frac{1}{2}4\sqrt{2}x^{-\frac{1}{2}} = \frac{2\sqrt{2}}{x^{\frac{1}{2}}}$$

At $x = 2$, $\frac{dy}{dx} = \frac{2\sqrt{2}}{\sqrt{2}} = 2$

Using $y - y_1 = m(x - x_1)$, the tangent to C at P is

$$y - 8 = 2(x - 2) = 2x - 4$$

$$y = 2x + 4 \quad \ldots\ldots \quad ❶$$

Q is on the lower half of the parabola where $y = -4\sqrt{2}x^{\frac{1}{2}}$

On the lower half of the parabola, in the fourth quadrant, the y-coordinates of P are negative.

$$\frac{dy}{dx} = -\frac{1}{2}4\sqrt{2}x^{-\frac{1}{2}} = -\frac{2\sqrt{2}}{x^{\frac{1}{2}}}$$

At $x = 32$, $\frac{dy}{dx} = -\frac{2\sqrt{2}}{\sqrt{32}} = -\frac{2\sqrt{2}}{4\sqrt{2}} = -\frac{1}{2}$

Using $y - y_1 = m(x - x_1)$, the tangent to C at Q is

$$y + 32 = -\frac{1}{2}(x - 32) = -\frac{1}{2}x + 16$$

$$y = -\frac{1}{2}x - 16 \quad \ldots\ldots \quad ❷$$

To find the x-coordinate of the intersection of the tangents, from ❶ and ❷

$$2x + 4 = -\frac{1}{2}x - 16$$

$$\frac{5}{2}x = -20 \Rightarrow x = -20\times\frac{2}{5} = -8$$

The equation of the directrix is $x = -8$ and, hence, the intersection of the tangents lies on the directrix.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 1

Question:

$\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$

Determine whether or not the following products exist. Where the product exists, evaluate the product. Where the product does not exist, give a reason for this.

a AB

b BA

c BAC

d CBA.

Solution:

a **AB** does not exist.
The matrix **A** is a 2×3 matrix.
The matrix **B** is a 2×2 matrix.
The number of columns in **A**, 3, is not equal to the number of rows in **B**, 2.

An $n\times m$ matrix can be multiplied by a $m\times p$ matrix. The number of columns in the left hand matrix must equal the number of rows in the right hand matrix.

b
$$\mathbf{BA}=\begin{pmatrix}2 & 0\\ 3 & -1\end{pmatrix}\begin{pmatrix}3 & 2 & 1\\ 0 & 2 & -1\end{pmatrix}$$
$$=\begin{pmatrix}2\times3+0\times0 & 2\times2+0\times2 & 2\times1+0\times(-1)\\ 3\times3+(-1)\times0 & 2\times3+(-1)\times2 & 3\times1+(-1)\times(-1)\end{pmatrix}$$
$$=\begin{pmatrix}6+0 & 4+0 & 2-0\\ 9+0 & 6-2 & 3+1\end{pmatrix}=\begin{pmatrix}6 & 4 & 2\\ 9 & 4 & 4\end{pmatrix}$$

c
$$\mathbf{BAC}=(\mathbf{BA})\mathbf{C}=\begin{pmatrix}6 & 4 & 2\\ 9 & 4 & 4\end{pmatrix}\begin{pmatrix}4\\ -3\\ 1\end{pmatrix}$$
$$=\begin{pmatrix}6\times4+4\times(-3)+2\times1\\ 9\times4+4\times(-3)+4\times1\end{pmatrix}$$
$$=\begin{pmatrix}24-12+2\\ 36-12+4\end{pmatrix}=\begin{pmatrix}14\\ 28\end{pmatrix}$$

As matrix multiplication is associative, you could work out $\mathbf{B}(\mathbf{AC})$ or $(\mathbf{BA})\mathbf{C}$ - they will give the same result. It is sensible to work out $(\mathbf{BA})\mathbf{C}$ as you have already worked out **BA** in part (b).

d **CBA** does not exist.
$\mathbf{CBA}=\mathbf{C}(\mathbf{BA})$
The matrix **C** is a 3×1 matrix.
The matrix **BA** is a 2×3 matrix.
The number of columns in **C**, 1, is not equal to the number of rows in **BA**, 2.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 2

Question:

$\mathbf{M} = \begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}$, $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Find the values of the constants a and b such that $\mathbf{M}^2 + a\mathbf{M} + b\mathbf{I} = \mathbf{O}$.

Solution:

$$\mathbf{M}^2 = \begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0\times 0+3\times(-1) & 0\times 3+3\times 2 \\ (-1)\times 0+2\times(-1) & (-1)\times 3+2\times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0-3 & 0+6 \\ 0-2 & -3+4 \end{pmatrix} = \begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix}$$

$\mathbf{M}^2$ is quite complicated to work out and it is sensible to calculate this before working out $\mathbf{M}^2 + a\mathbf{M} + b\mathbf{I}$.

$$\mathbf{M}^2 + a\mathbf{M} + b\mathbf{I} = \mathbf{O}$$

$$\begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix} + a\begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3a \\ -a & 2a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -3+b & 6+3a \\ -2-a & 1+2a+b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Equating the top left elements

$$-3+b=0 \Rightarrow b=3$$

Equating the top right elements

$$6+3a=0 \Rightarrow a=-2$$

$$a=-2,\ b=3$$

There are four elements which could be equated but you only need to equate two of them to find a and b. You could use the others to check your working. For example; if $a=-2$, $b=3$ then $1+2a+b=1-4+3$ which does equal 0.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 3

Question:

$\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix}$

Show that $\mathbf{A}^2 - 10\mathbf{A} + 21\mathbf{I} = \mathbf{O}$.

Solution:

$$\mathbf{A}^2 = \begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 4\times4+1\times3 & 4\times1+1\times6 \\ 3\times4+6\times3 & 3\times1+6\times6 \end{pmatrix}$$

$$= \begin{pmatrix} 16+3 & 4+6 \\ 12+18 & 3+36 \end{pmatrix} = \begin{pmatrix} 19 & 10 \\ 30 & 39 \end{pmatrix}$$

$$\mathbf{A}^2 - 10\mathbf{A} + 21\mathbf{I} = \begin{pmatrix} 19 & 10 \\ 30 & 39 \end{pmatrix} - 10\begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix} + 21\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 19 & 10 \\ 30 & 39 \end{pmatrix} - \begin{pmatrix} 40 & 10 \\ 30 & 60 \end{pmatrix} + \begin{pmatrix} 21 & 0 \\ 0 & 21 \end{pmatrix}$$

$$= \begin{pmatrix} 19-40+21 & 10-10+0 \\ 30-30+0 & 39-60+21 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$= \mathbf{O}$, as required.

To show the matrix relation given in the question, you must work on the left hand side of the equation $\mathbf{A}^2 - 10\mathbf{A} + 21\mathbf{I}$ and manipulate it until you get the right hand side of the equation $\mathbf{O}$.

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Review Exercise
Exercise A, Question 4

Question:

$\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Find an expression for λ, in terms of a, b, c and d, so that $\mathbf{A}^2-(a+d)\mathbf{A}=\lambda\mathbf{I}$, where $\mathbf{I}$ is the 2×2 unit matrix.

Solution:

$$\mathbf{A}^2=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$$

$$\mathbf{A}^2-(a+d)\mathbf{A}$$
$$=\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}-\begin{pmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{pmatrix}$$
$$=\begin{pmatrix} a^2+bc-a^2-ad & ab+bd-ab-bd \\ ac+cd-ac-ad & bc+d^2-ad-d^2 \end{pmatrix}$$
$$=\begin{pmatrix} bc-ad & 0 \\ 0 & bc-ad \end{pmatrix}=\lambda\mathbf{I}=\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$\mathbf{I}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so

$\lambda\mathbf{I}=\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

You can write down the results of simple calculations like this without showing all of the working.

Equating the top left (or bottom right elements)

$$\lambda=bc-ad$$

Note that $\lambda=-\det(\mathbf{A})$.

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Review Exercise
Exercise A, Question 5

Question:

$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ p & -1 \end{pmatrix}$, where p is a real constant. Given that $\mathbf{A}$ is singular,

a find the value of p.

Given instead that det ($\mathbf{A}$) = 4,

b find the value of p.

Using the value of p found in **b**,

c show that $\mathbf{A}^2 - \mathbf{A} = k\mathbf{I}$, stating the value of the constant k.

Solution:

a $\det(\mathbf{A}) = 2\times(-1) - 3\times p = -2-3p$

If $\mathbf{A}$ is singular, $\det(\mathbf{A}) = 0$.

$-2-3p = 0 \Rightarrow 3p = -2 \Rightarrow p = -\frac{2}{3}$

You need to memorise that, if $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

b As in part (a), $\det(\mathbf{A}) = -2-3p$

$-2-3p = 4 \Rightarrow -3p = 6 \Rightarrow p = -2$

c $\mathbf{A}^2 = \begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 4-6 & 6-3 \\ -4+2 & -6+1 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -2 & -5 \end{pmatrix}$

$\mathbf{A}^2 - \mathbf{A} = \begin{pmatrix} -2 & 3 \\ -2 & -2 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}$

$= \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = -4\mathbf{I}$

This is the required result with $k = -4$.

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Review Exercise
Exercise A, Question 6

Question:

$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix}$

a Find $\mathbf{A}^{-1}$.

Given that $\mathbf{A}^5 = \begin{pmatrix} 251 & -109 \\ -327 & 142 \end{pmatrix}$,

b find $\mathbf{A}^4$.

Solution:

a

$$\det(\mathbf{A}) = 2\times1-(-1)\times(-3)2-3=-1$$

$$\text{If } \mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } \mathbf{A}^{-1}=\frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

$$\mathbf{A}^{-1}=\frac{1}{-1}\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}=\begin{pmatrix} -1 & -1 \\ -3 & -2 \end{pmatrix}$$

You should know this formula. It is to your advantage to quote the formula in your solution. If you make a mistake, the examiner will know that you are trying to do the right thing.

b

$$\mathbf{A}^4\mathbf{A}=\mathbf{A}^5$$

$$\mathbf{A}^4\mathbf{A}\mathbf{A}^{-1}=\mathbf{A}^5\mathbf{A}^{-1}$$

$$\mathbf{A}^4=\mathbf{A}^5\mathbf{A}^{-1}=\begin{pmatrix} 251 & -109 \\ -327 & 142 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -3 & -2 \end{pmatrix}$$

$$=\begin{pmatrix} -251+327 & -251+218 \\ 327-426 & 327-284 \end{pmatrix}=\begin{pmatrix} 76 & -33 \\ -99 & 43 \end{pmatrix}$$

It is much quicker to multiply $\mathbf{A}^5$ by $\mathbf{A}^{-1}$ than to repeatedly multiply $\mathbf{A}$ by itself. For whole numbers, the ordinary algebraic rules for indices apply to matrices and it will help you if you remember this.

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Review Exercise
Exercise A, Question 7

Question:

A triangle T, of area 18cm^2, is transformed into a triangle T' by the matrix **A** where, $\mathbf{A} = \begin{pmatrix} k & k-1 \\ -3 & 2k \end{pmatrix}$, $k \in \mathbb{R}$.

a Find det (**A**), in terms of k.

Given that the area of T' is $198\ \text{cm}^2$,

b find the possible values of k.

Solution:

a If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

$$\det(\mathbf{A}) = k \times 2k - (k-1) \times (-3)$$
$$= 2k^2 + 3k - 3$$

b The triangle has been enlarged by a factor of

$$\frac{198}{11} = 11$$

So $\det(\mathbf{A}) = 11$

$$2k^2 + 3k - 3 = 11$$
$$2k^2 + 3k - 14 = (2k+7)(k-2) = 0$$
$$k = -\frac{7}{2}, 2$$

The determinant is the area scale factor in transformations. This is equivalent to $\frac{\text{area of image}}{\text{area of object}} = \det(\mathbf{A})$. So the scale factor in part (a) must equal the determinant in part (b).

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Review Exercise
Exercise A, Question 8

Question:

A linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\mathbf{p} = \mathbf{Nq}$, where $\mathbf{N}$ is a 2×2 matrix and $\mathbf{p}$, $\mathbf{q}$ are 2×1 column vectors.

Given that $\mathbf{p} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$ when $\mathbf{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and that $\mathbf{p} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$ when $\mathbf{q} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, find $\mathbf{N}$.

Solution:

Let $\mathbf{N} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

You need to introduce some algebraic variables to help you to obtain equations. You can use any symbols you like for the elements of the matrix, except for those already used in the question.

$$\mathbf{p} = \mathbf{Nq}$$

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

Equating elements

$$a = 3, c = 7$$

$$\begin{pmatrix} 6 \\ -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2a - 3b \\ 2c - 3d \end{pmatrix}$$

You use the values for a and c which you found earlier.

Equating the upper elements

$$2a - 3b = 6$$

$$a = 3 \Rightarrow 6 - 3b = 6 \Rightarrow b = 0$$

Equating the lower elements

$$2c - 3d = -1$$

$$c = 7 \Rightarrow 14 - 3d = -1 \Rightarrow 3d = 15 \Rightarrow d = 5$$

$$\mathbf{N} = \begin{pmatrix} 3 & 0 \\ 7 & 5 \end{pmatrix}$$

Solutionbank FP1
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Review Exercise
Exercise A, Question 9

Question:

$\mathbf{A} = \begin{pmatrix} 4 & -1 \\ -6 & 2 \end{pmatrix}$, $\mathbf{B}^{-1} = \begin{pmatrix} 2 & 0 \\ 3 & p \end{pmatrix}$

a Find $\mathbf{A}^{-1}$.

b Find $(\mathbf{AB})^{-1}$, in terms of p.

Given also that $\mathbf{AB} = \begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix}$,

c find the value of p.

Solution:

a If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{A}^{-1} = \dfrac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$\det(\mathbf{A}) = 4 \times 2 - (-1) \times (-6) = 8 - 6 = 2$

$$\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{pmatrix}$$

b $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

You need to remember this property of the inverse of matrices. The order of **A** and **B** is reversed in this formula.

$$= \begin{pmatrix} 2 & 0 \\ 3 & p \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 3p+3 & 2p+\frac{3}{2} \end{pmatrix}$$

c $(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{I}$

The product of any matrix and its inverse is **I**. This applies to a product matrix, **AB** in this case, as well as to a matrix such as **A**.

$$\begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3p+3 & 2p+\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Equating the upper left elements

Finding all four of the elements of the product matrix of the left hand side of this equation would be lengthy. To find p, you only need one equation, so you only need to consider one element. Here the upper left hand element has been used but you could choose any of the four elements.

$$-1 \times 2 + 2(3p+3) = 1$$

$$-2 + 6p + 6 = 1$$

$$6p = -3$$

$$p = -\frac{1}{2}$$

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Review Exercise
Exercise A, Question 10

Question:

$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}$

a Show that $\mathbf{A}^3 = \mathbf{I}$.

b Deduce that $\mathbf{A}^2 = \mathbf{A}^{-1}$.

c Use matrices to solve the simultaneous equations

$$\begin{aligned} 2x - y &= 3, \\ 7x - 3y &= 2. \end{aligned}$$

Solution:

a $\mathbf{A}^2 = \begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}$

$= \begin{pmatrix} 4-7 & -2+3 \\ 14-21 & -7+9 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -7 & 2 \end{pmatrix}$

$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A}$

$= \begin{pmatrix} -3 & 1 \\ -7 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}$

$= \begin{pmatrix} -6+7 & 3-3 \\ -14+14 & 7-6 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$, as required.

> $\mathbf{A}^3 = \mathbf{AAA} = \mathbf{A}^2\mathbf{A}$. Work out $\mathbf{A}^2$ first and then multiply the result by $\mathbf{A}$.

b $\mathbf{A}^3 = \mathbf{I}$

Multiply both sides by $\mathbf{A}^{-1}$

$\mathbf{A}^3\mathbf{A}^{-1} = \mathbf{IA}^{-1}$

$\mathbf{A}^2 = \mathbf{A}^{-1}$, as required.

> It helps if you remember that, for whole numbers, the ordinary algebraic rules for indices apply to matrices. In more detail;
> $\mathbf{A}^3\mathbf{A}^{-1} = \left(\mathbf{A}^2\mathbf{A}\right)\mathbf{A}^{-1} = \mathbf{A}^2\left(\mathbf{AA}^{-1}\right) = \mathbf{A}^2\mathbf{I} = \mathbf{A}^2$

c Writing the simultaneous equations as matrices

$$\begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Multiply both sides of this equation on the left by $\mathbf{A}^{-1}$, which, in this case, is $\mathbf{A}^2$.

$$\mathbf{A}^2\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^2\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\mathbf{A}^3\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{I}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -7 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -9+2 \\ -21+4 \end{pmatrix} = \begin{pmatrix} -7 \\ -17 \end{pmatrix}$$

> To solve simultaneous equations using matrices, you need to multiply both sides of a matrix equation by the appropriate inverse matrix. In this question, part (b) has shown that the inverse matrix is $\mathbf{A}^2$ and, as you worked this out in part (a), there is no need to work the inverse matrix out again.

Equating elements

$$x = -7,\ y = -17$$

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Review Exercise
Exercise A, Question 11

Question:

$\mathbf{A} = \begin{pmatrix} 5 & -2 \\ 5 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}$

a Find $\mathbf{A}^{-1}$.

b Show that $\mathbf{A}^{-1}\mathbf{BA} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, stating the values of the constants λ_1 and λ_2.

Solution:

a $\det(\mathbf{A}) = 5 \times 5 - 5 \times (-2) = 25 + 10 = 35$

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$$\mathbf{A}^{-1} = \frac{1}{35}\begin{pmatrix} 5 & 2 \\ -5 & 5 \end{pmatrix}$$

This could be written as $\begin{pmatrix} \frac{1}{7} & \frac{2}{35} \\ -\frac{1}{7} & \frac{1}{7} \end{pmatrix}$.

Either form is acceptable.

b $\mathbf{A}^{-1}\mathbf{BA} = \mathbf{A}^{-1}(\mathbf{BA})$

$$= \mathbf{A}^{-1}\begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ 5 & 5 \end{pmatrix}$$

$$= \mathbf{A}^{-1}\begin{pmatrix} 20+10 & -8+10 \\ 25+5 & -10+5 \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 30 & 2 \\ 30 & -5 \end{pmatrix}$$

$$= \frac{1}{35}\begin{pmatrix} 5 & 2 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} 30 & 2 \\ 30 & -5 \end{pmatrix}$$

$$= \frac{1}{35}\begin{pmatrix} 150+60 & 10-10 \\ -150+150 & -10-25 \end{pmatrix}$$

$$= \frac{1}{35}\begin{pmatrix} 210 & 0 \\ 0 & -35 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & -1 \end{pmatrix}$$

This is the required form with $\lambda_1 = 6$ and $\lambda_2 = -1$.

As matrix multiplication is associative, you could work out this triple product as $(\mathbf{A}^{-1}\mathbf{B})\mathbf{A}$ but $\mathbf{A}^{-1}$ has an awkward fraction, so it is sensible to evaluate $\mathbf{BA}$ first.

If you go on to study the FP3 module, you will learn how to carry out calculations like this with larger matrices. These calculations have important applications to physics and statistics.

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Review Exercise
Exercise A, Question 12

Question:

$\mathbf{A} = \begin{pmatrix} 4p & -q \\ -3p & q \end{pmatrix}$, where p and q are non-zero constants.

a Find $\mathbf{A}^{-1}$, in terms of p and q.

Given that $\mathbf{AX} = \begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$,

b find $\mathbf{X}$, in terms of p and q.

Solution:

a $\det(\mathbf{A}) = 4p \times q - (-q) \times (-3p)$

$= 4pq - 3pq = pq$

$$\mathbf{A}^{-1} = \frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}$$

The alternative answer, multiplying the matrix by the scalar $\frac{1}{pq}$, $\begin{pmatrix} \frac{1}{p} & \frac{1}{p} \\ \frac{3}{q} & \frac{4}{q} \end{pmatrix}$ would be an equally good one.

b $\mathbf{AX} = \begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$

Multiply both sides on the left by $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$$

$$\mathbf{X} = \frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$$

$$= \frac{1}{pq}\begin{pmatrix} 2pq - pq & 3q^2 + q^2 \\ 6p^2 - 4p^2 & 9pq + 4pq \end{pmatrix}$$

$$= \frac{1}{pq}\begin{pmatrix} pq & 4q^2 \\ 2p^2 & 13pq \end{pmatrix}$$

It is important to multiply by $\mathbf{A}^{-1}$ on the correct side of the expression. As shown here, multiplying on the left of $\mathbf{AX}$, you get $\mathbf{A}^{-1}\mathbf{AX} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{IX} = \mathbf{X}$, which is what you are asked to find. On the right of $\mathbf{AX}$, you would get $\mathbf{AXA}^{-1}$, which does not simplify, and no further progress can be made. Working out $\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}\frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}$ instead of $\frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$ is a common error.

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Review Exercise
Exercise A, Question 13

Question:

$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -1 \\ -4 & 5 \end{pmatrix}$

Find

a **AB**,

b **AB** – **BA**.

Given that **C** = **AB** – **BA**,

c find $\mathbf{C}^2$,

d give a geometrical interpretation of the transformation represented by $\mathbf{C}^2$.

Solution:

a $\mathbf{AB} = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -4 & 5 \end{pmatrix}$

$= \begin{pmatrix} 12-8 & -4+10 \\ 15-12 & -5+15 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 3 & 10 \end{pmatrix}$

Matrix multiplication is not commutative and, as in this question, **AB** and **BA** can be quite different.

b $\mathbf{BA} = \begin{pmatrix} 3 & -1 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$

$= \begin{pmatrix} 12-5 & 6-3 \\ -16+25 & -8+15 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 9 & 7 \end{pmatrix}$

$\mathbf{AB} - \mathbf{BA} = \begin{pmatrix} 4 & 6 \\ 3 & 10 \end{pmatrix} - \begin{pmatrix} 7 & 3 \\ 9 & 7 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ -6 & 3 \end{pmatrix}$

c $\mathbf{C}^2 = \begin{pmatrix} -3 & 3 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} -3 & 3 \\ -6 & 3 \end{pmatrix} = \begin{pmatrix} 9-18 & -9+9 \\ 18-18 & -18+9 \end{pmatrix}$

$= \begin{pmatrix} -9 & 0 \\ 0 & -9 \end{pmatrix}$

For all $k \neq 0$, the matrix $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ represents an enlargement, centre $(0, 0)$, scale factor k.

d Enlargement, centre $(0, 0)$, scale factor -9

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Review Exercise
Exercise A, Question 14

Question:

The matrix **A** represents reflection in the x-axis.

The matrix **B** represents a rotation of $135°$, in the anti-clockwise direction, about (0, 0).

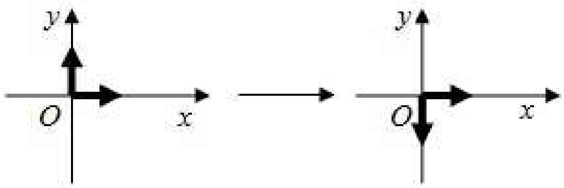
Given that **C** = **AB**,

a find the matrix **C**,

b show that $\mathbf{C}^2 = \mathbf{I}$.

Solution:

a Reflection in the x axis transforms



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix}$$

$(1, 0)$ lies on the x-axis and so is not changed by reflection in the x-axis.

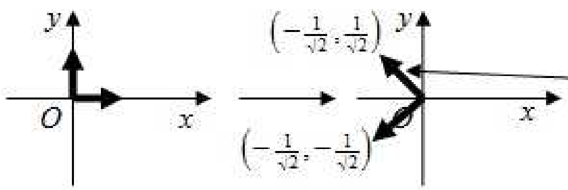
$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\-1\end{pmatrix}$$

So

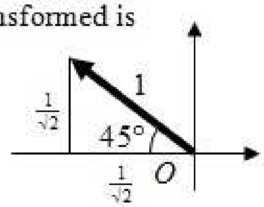
$$\mathbf{A} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

Unless the question states otherwise, it is acceptable to write down a simple matrix like this without working.

Rotation of $+135°$ about $(0, 0)$ transforms



The geometry of the vector to which $\begin{pmatrix}1\\0\end{pmatrix}$ is transformed is



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}-\dfrac{1}{\sqrt{2}}\\[2mm] \dfrac{1}{\sqrt{2}}\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}-\dfrac{1}{\sqrt{2}}\\[2mm] -\dfrac{1}{\sqrt{2}}\end{pmatrix}$$

Arrows have been added to this calculation so that you can see where the columns in **B** come from.

So

$$\mathbf{B} = \begin{pmatrix}-\dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}}\\[2mm] \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}}\end{pmatrix}$$

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

b $$\mathbf{C}^2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

As an example of the calculations;
$$-\frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} = +\frac{1\times 1}{\sqrt{2}\times\sqrt{2}} = \frac{1}{2}$$

$$= \begin{pmatrix} \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) & \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) \\ \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) & \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}\text{, as required.}$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 15

Question:

The linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix **M**, where $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The transformation T maps the point with coordinates (1, 0) to the point with coordinates (3, 2) and the point with coordinates (2, 1) to the point with coordinates (6, 3).

a Find the values of a, b, c and d.

b Show that $\mathbf{M}^2 = \mathbf{I}$.

The transformation T maps the point with coordinates (p, q) to the point with coordinates $(8, -3)$.

c Find the value of p and the value of q.

Solution:

a $\mathbf{M}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$

$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}a\\c\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$

Equating the elements

$a=3, c=2$

$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2a+b\\2c+d\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$

Equating the upper elements

$2a+b=2$

$a=3 \Rightarrow 6+b=2 \Rightarrow b=-4$

Equating the lower elements

$2c+d=1$

$c=2 \Rightarrow 4+d=1 \Rightarrow d=-3$

$a=3, b=-4, c=2, d=-3$

In questions about transformations, you need to write the coordinates of points as column vectors. For example, the coordinate $(1, 0)$ is written as the column vector $\begin{pmatrix}1\\0\end{pmatrix}$.

b $\mathbf{M}=\begin{pmatrix}3 & -4\\2 & -3\end{pmatrix}$

$\mathbf{M}^2=\begin{pmatrix}3 & -4\\2 & -3\end{pmatrix}\begin{pmatrix}3 & -4\\2 & -3\end{pmatrix}=\begin{pmatrix}9-8 & -12+12\\6-6 & -8+9\end{pmatrix}$

$=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\mathbf{I}$, as required

c As $\mathbf{M}^2=\mathbf{I}$, $\mathbf{M}^{-1}\mathbf{M}^2=\mathbf{M}^{-1}\mathbf{I}$

$\mathbf{M}=\mathbf{M}^{-1}$

The matrix **M** is its own inverse. This follows from the result in part (b). In more detail; $\mathbf{M}^2=\mathbf{I}$

$\mathbf{MM}=\mathbf{I}$

$\mathbf{M}^{-1}(\mathbf{MM})=\mathbf{M}^{-1}\mathbf{I}$

$(\mathbf{M}^{-1}\mathbf{M})\mathbf{M}=\mathbf{M}^{-1}$

$\mathbf{IM}=\mathbf{M}^{-1}$

$\mathbf{M}=\mathbf{M}^{-1}$

$\mathbf{M}\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}8\\-3\end{pmatrix}$

$\mathbf{M}^{-1}\mathbf{M}\begin{pmatrix}p\\q\end{pmatrix}=\mathbf{M}^{-1}\begin{pmatrix}8\\-3\end{pmatrix}$

$\mathbf{I}\begin{pmatrix}p\\q\end{pmatrix}=\mathbf{M}\begin{pmatrix}8\\-3\end{pmatrix}$

In this question, as **M** is its own inverse, you can replace $\mathbf{M}^{-1}$ by **M**.

$\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}3 & -4\\2 & -3\end{pmatrix}\begin{pmatrix}8\\-3\end{pmatrix}=\begin{pmatrix}24+12\\16+9\end{pmatrix}=\begin{pmatrix}36\\25\end{pmatrix}$

Hence $p=36, q=25$

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Review Exercise
Exercise A, Question 16

Question:

16 The linear transformation T is defined by $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2y - x \\ 3y \end{pmatrix}$.

The linear transformation T is represented by the matrix **C**.

a Find **C**.

The quadrilateral $OABC$ is mapped by T to the quadrilateral $OA'B'C'$, where the coordinates of A', B' and C' are (0, 3), (10, 15) and (10, 12) respectively.

b Find the coordinates of A, B and C.

c Sketch the quadrilateral $OABC$ and verify that $OABC$ is a rectangle.

Solution:

a $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2y-x \\ 3y \end{pmatrix} = \begin{pmatrix} -1x+2y \\ 0x+3y \end{pmatrix}$

$= \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

So $\mathbf{C} = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$

b $\det(\mathbf{C}) = -1\times 3 - 3\times 0 = -3$

$\mathbf{C}^{-1} = \frac{1}{-3}\begin{pmatrix} 3 & -2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$

You are given the results of transforming the points by T and are asked to find the original points. You are "working backwards" to the original points and you will need the inverse matrix.

Let the coordinates of A, B and C be (x_A, y_A), (x_B, y_B) and (x_C, y_C) respectively.

$\mathbf{C}\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$

$\mathbf{C}^{-1}\mathbf{C}\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \mathbf{C}^{-1}\begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$

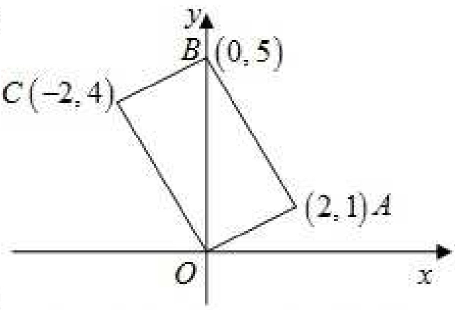
$\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \begin{pmatrix} -1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$

$= \begin{pmatrix} 2 & -10+10 & -10+8 \\ 1 & 5 & 4 \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 & -2 \\ 1 & 5 & 4 \end{pmatrix}$

Hence $A:(2,1)$, $B:(0,5)$, $C:(-2,4)$

c



d

Considering the gradients of the sides

$m_{OA} = \frac{1}{2}$; $m_{CB} = \frac{5-4}{0-(-2)} = \frac{1}{2}$

So OA is parallel to CB.

$m_{OC} = \frac{4-0}{-2-0} = -2$; $m_{AB} = \frac{5-1}{0-2} = \frac{4}{-2} = -2$

Using the properties of quadrilaterals you learnt for GCSE, there are many alternative ways of showing that $OABC$ is a rectangle. This is just one of many possibilities, using the result you learnt in the C1 module that the gradient of the line joining (x_1, y_1) to (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

So OC is parallel to AB.
The opposite sides of $OABC$ are parallel to each other and so $OABC$ is a parallelogram.

Also $m_{OA} \times m_{OC} = \frac{1}{2} \times -2 = -1$.

So OA is perpendicular to OC.
So the parallelogram $OABC$ contains a right angle and, hence, $OABC$ is a right angle.

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Review Exercise
Exercise A, Question 17

Question:

$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0.8 & -0.4 \\ 0.2 & -0.6 \end{pmatrix}$ and $\mathbf{C} = \mathbf{AB}$.

a Find **C**.

b Give a geometrical interpretation of the transformation represented by **C**.

The square *OXYZ*, where the coordinates of *X* and *Y* are (0, 3) and (3, 3), is transformed into the quadrilateral $OX'Y'Z'$, by the transformation represented by **C**.

c Find the coordinates of Z'.

Solution:

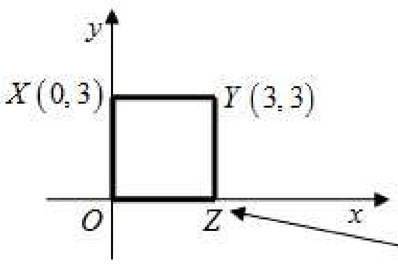
a
$$\mathbf{C} = \mathbf{BA} = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 0.8 & -0.4 \\ 0.2 & -0.6 \end{pmatrix}$$
$$= \begin{pmatrix} 2.4-0.4 & -1.2+1.2 \\ -0.8+0.8 & 0.4-2.4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

b
$$\mathbf{C} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

So the transformation can be interpreted as reflection in the x-axis followed by an enlargement, centre $(0, 0)$, scale factor 2.

These transformations have the same effect with their order reversed so "enlargement, centre $(0, 0)$, scale factor 2 followed by reflection in the x-axis" is an equally good answer.

c



You have been asked to find Z' - that is the point to which Z is transformed. You have not been given the coordinates of Z. Drawing a quick sketch makes it clear that Z has coordinates $(3, 0)$.

The coordinates of Z are $(3, 0)$.

To find the coordinates of Z'

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

The coordinates of Z' are $(6, 0)$.

Alternatively you can argue, using the answer to part (b), that reflecting $(3, 0)$ in the x-axis leaves the point unchanged as it lies on the x-axis. An enlargement of scale factor 2 then leads to $(6, 0)$.

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Review Exercise
Exercise A, Question 18

Question:

Given that $\mathbf{A} = \begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, find the matrices **C** and **D** such that

a $\mathbf{AC} = \mathbf{B}$,

b $\mathbf{DA} = \mathbf{B}$.

A linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ is defined by the matrix **B**.

c Prove that the line with equation $y = mx$ is mapped onto another line through the origin O under this transformation.

d Find the gradient of this second line in terms of m.

Solution:

a
$$\mathbf{AC} = \mathbf{B}$$
$$\mathbf{A}^{-1}\mathbf{AC} = \mathbf{A}^{-1}\mathbf{B}$$
So $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$

$\det(\mathbf{A}) = 5\times(-1) - 3\times(-2) = -5+6 = 1$

$$\mathbf{A}^{-1} = \frac{1}{1}\begin{pmatrix}-1 & -3\\ 2 & 5\end{pmatrix} = \begin{pmatrix}-1 & -3\\ 2 & 5\end{pmatrix}$$

$$\mathbf{C} = \mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix}-1 & -3\\ 2 & 5\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix} = \begin{pmatrix}-1 & -1-6\\ 2 & 2+10\end{pmatrix}$$
$$= \begin{pmatrix}-1 & -7\\ 2 & 12\end{pmatrix}$$

$\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{D} = \mathbf{BA}^{-1}$ but, as matrix multiplication is not commutative, $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{BA}^{-1}$ are different. You must be careful of the order in which you multiply matrices.

b
$$\mathbf{DA} = \mathbf{B}$$
$$\mathbf{DAA}^{-1} = \mathbf{BA}^{-1}$$
So $\mathbf{D} = \mathbf{BA}^{-1}$

$$\mathbf{D} = \mathbf{BA}^{-1} = \begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}-1 & -3\\ 2 & 5\end{pmatrix} = \begin{pmatrix}-1+2 & -3+5\\ 4 & 10\end{pmatrix}$$
$$= \begin{pmatrix}1 & 2\\ 4 & 10\end{pmatrix}$$

c Let the general point on $y = mx$ have coordinates (t, mt)

If $x = t$, as $y = mx$, then $y = mt$. The variable t is being used as a parameter. You used parameters in Chapter 3 to solve questions involving parabolas and rectangular hyperbolas.

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}t\\ mt\end{pmatrix} = \begin{pmatrix}t+mt\\ 2mt\end{pmatrix}$$

Equating the elements of the 2×1 matrices
$$x = t + mt,\ y = 2mt$$
$$y = 2mt \Rightarrow t = \frac{y}{2m}$$

Substituting into the equation for x
$$x = \frac{y}{2m} + m\times\frac{y}{2m} = \frac{y}{2m}(1+m)$$

Making y the subject of the formula
$$y = \frac{2m}{1+m}x$$

Eliminating t between these two expressions gives a linear equation relating y and x. The equation has no x^2, y^2, xy or higher powered terms. Therefore, the equation represents a straight line.

Comparing with the standard form of a line, $y = mx + c$,
$c = 0$, the line goes through the origin.
The line with equation $y = mx$ is transformed to another line passing through O.

d The gradient of this second line is $\dfrac{2m}{1+m}$.

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Review Exercise
Exercise A, Question 19

Question:

Referred to an origin O and coordinate axes Ox and Oy, transformations from $\mathbb{R}^2 \to \mathbb{R}^2$ are represented by the matrices **L**, **M** and **N**, where

$\mathbf{L} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{M} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

a Explain the geometrical effect of the transformations **L** and **M**.

b Show that $\mathbf{LM} = \mathbf{N}^2$.

The transformation represented by the matrix **N** consists of a rotation of angle θ about O, followed by an enlargement, centre O, with positive scale factor k.

c Find the value of θ and the value of k.

d Find $\mathbf{N}^8$.

Solution:

a **L** represents rotation through $90°$, anti-clockwise, about the origin O.
M represents an enlargement, centre O, scale factor 2.

If you do not specify anticlockwise, positive angles are, conventionally, taken as anticlockwise and negative angles as clockwise. So, in this case, if you omitted "anti-clockwise", you would still be correct. Often $+90°$ is written to emphasize that the angle is anti-clockwise.

b $\mathbf{LM} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

$\mathbf{N}^2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1-1 & -1-1 \\ 1+1 & -1+1 \end{pmatrix}$

$= \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

So $\mathbf{LM} = \mathbf{N}^2$, they are both equal to $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

c The result of part (b) can be interpreted as showing that the transformation represented by **N** <u>applied twice</u> is equivalent to rotation through $+90°$ about O followed by an enlargement, centre O, scale factor 2.
So the transformation represented by **N** <u>applied once</u> is equivalent to rotation through $+45°$ about O followed by an enlargement, centre O, scale factor $\sqrt{2}$.
$\theta = +45°$, $k = \sqrt{2}$.

Alternatively, it is possible to solve part (c) using matrices. The matrix representing a rotation of $+45°$ about O is $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ and the critical step is showing that

$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \mathbf{N}$

.

d $\mathbf{N}^8$ represents the transformation represented by **N** <u>applied eight times.</u> This will rotate about the origin $8 \times 45° = 360°$ (which is the identity transformation), followed by an enlargement, centre O, scale factor $(\sqrt{2})^8 = 16$.

Hence $\mathbf{N}^8 = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$.

Again, this can be done by matrices. You already know $\mathbf{N}^2$ from part (b) and you could then use

$\mathbf{N}^4 = \mathbf{N}^2\mathbf{N}^2$

and $\mathbf{N}^8 = \mathbf{N}^4\mathbf{N}^4$

to reach $\mathbf{N}^8$. Unless a question specifies a particular method, any correct alternative method can be used.

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Review Exercise
Exercise A, Question 20

Question:

A, **B** and **C** are 2×2 matrices.

a Given that **AB** = **AC**, and that **A** is not singular, prove that **B** = **C**.

b Given that **AB** = **AC**, where $\mathbf{A} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$, find a matrix **C** whose elements are all non-zero.

Solution:

a $\mathbf{AB} = \mathbf{AC}$

Multiplying both sides on the left by $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1}(\mathbf{AB}) = \mathbf{A}^{-1}(\mathbf{AC})$$

As matrices are associative

$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{C}$$

Using $\mathbf{AA}^{-1} = \mathbf{I}$

$$\mathbf{IB} = \mathbf{IC}$$

As the identity matrix does not change another matrix

$\mathbf{B} = \mathbf{C}$, as required.

When you are asked to prove a result, you must give each essential step in the argument. Your argument here must include multiplying by $\mathbf{A}^{-1}$ on the left of both sides of the equation and showing where you use $\mathbf{AA}^{-1} = \mathbf{I}$.

b $\mathbf{AB} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 21 \\ 1 & 7 \end{pmatrix}$

Let $\mathbf{C} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\mathbf{AC} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3a+6c & 3b+6d \\ a+2c & b+2d \end{pmatrix}$

$$\mathbf{AB} = \mathbf{AC}$$

$$\begin{pmatrix} 3 & 21 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 3a+6c & 3b+6d \\ a+2c & b+2d \end{pmatrix}$$

$\det(\mathbf{A}) = 6 - 6 = 0$ so $\mathbf{A}$ is singular. This question is a good example of the difficulties which can arise with non-singular matrices. There is no inverse matrix $\mathbf{A}^{-1}$ here and so you cannot use the usual rules of matrix algebra to remove $\mathbf{A}$.

Equating the upper left elements

$$3 = 3a + 6c \quad \ldots \ldots ❶$$

Equating the lower left elements

$$1 = a + 2c \quad \ldots \ldots ❷$$

❶ and ❷ are the same equation (❶ is ❷ $\times 3$)

Apart from the condition that the elements are non-zero, there is a free choice of a.

Let $a = 3$, then substituting in ❷,

$1 = 3 + 2c \Rightarrow c = -1$

This pair of equations are satisfied by infinitely many pairs of numbers. You just have to choose any two non-zero numbers which satisfy them. For example, $a = -1, c = 1$ would do just as well.

Equating the upper right elements

$$21 = 3b + 6d \quad \ldots \ldots ❸$$

Equating the lower right elements

$$7 = b + 2d \quad \ldots \ldots ❹$$

❸ and ❹ are the same equation (❸ is ❹ $\times 3$)

Apart from the condition that the elements are non-zero, there is a free choice of b.

Let $b = 1$, then substituting in ❷,

$7 = 1 + 2d \Rightarrow d = 3$

$$\mathbf{C} = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$$

Check:

$\mathbf{AC} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 21 \\ 1 & 7 \end{pmatrix} = \mathbf{AB}$, as required.

This is an unusual question and it is a good idea to check that your answer does give the correct result. You may well have a different $\mathbf{C}$ from that shown here, but you can check your answer by finding $\mathbf{AC}$. If you obtain $\begin{pmatrix} 3 & 21 \\ 1 & 7 \end{pmatrix}$, your answer is correct.

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Review Exercise
Exercise A, Question 21

Question:

Use standard formulae to show that $\sum_{r=1}^{n} 3r(r-1) = n(n^2-1)$.

Solution:

$$\sum_{r=1}^{n} 3r(r-1) = \sum_{r=1}^{n}(3r^2 - 3r)$$

$$= 3\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r$$

$$= \frac{\cancel{3}^1 n(n+1)(2n+1)}{\cancel{6}^2} - \frac{3n(n+1)}{2}$$

$$= \frac{n(n+1)(2n+1)}{2} - \frac{3n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}[(2n+1)-3]$$

$$= \frac{n(n+1)(2n-2)}{2}$$

$$= \frac{n(n+1)\cancel{2}(n-1)}{\cancel{2}}$$

$$= n(n^2-1), \text{ as required.}$$

Multiply out the brackets and write the expression in terms of $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$.

You can then use the standard formulae.

After "cancelling" the fractions, look for the common factors in your expressions, here shown in **bold**;

$$\frac{\boldsymbol{n(n+1)}(2n+1)}{2} - \frac{3\boldsymbol{n(n+1)}}{2}.$$

These, with the 2, are then taken outside a bracket;

$$\frac{\boldsymbol{n(n+1)}}{2}[(2n+1)-3].$$

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Review Exercise
Exercise A, Question 22

Question:

Use standard formulae to show that $\sum_{r=1}^{n}(r^2-1) = \frac{n}{6}(2n+5)(n-1)$.

Solution:

$$\sum_{r=1}^{n}(r^2-1) = \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} 1$$

$$= \frac{n(n+1)(2n+1)}{6} - n$$

$$= \frac{n(n+1)(2n+1)}{6} - \frac{6n}{6}$$

$$= \frac{n}{6}\left[(n+1)(2n+1)-6\right]$$

$$= \frac{n}{6}\left[2n^2+3n+1-6\right]$$

$$= \frac{n}{6}\left[2n^2+3n-5\right]$$

$$= \frac{n}{6}(2n+5)(n-1), \text{ as required.}$$

$$\sum_{r=1}^{n} 1 = \underbrace{1+1+1+\ \dots\ +1}_{n \text{ times}} = n$$

It is a common error to write $\sum_{r=1}^{n} 1 = 1$.

You put both terms over a common denominator, here 6.

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Review Exercise
Exercise A, Question 23

Question:

Use standard formulae to show that $\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(4n^2-1)$.

Solution:

$$\sum_{r=1}^{n}(2r-1)^2 = \sum_{r=1}^{n}(4r^2-4r+1)$$

$$= 4\sum_{r=1}^{n}r^2 - 4\sum_{r=1}^{n}r + \sum_{r=1}^{n}1$$

$$= \frac{\cancel{4}^2 n(n+1)(2n+1)}{\cancel{6}^3} - \frac{\cancel{4}^2 n(n+1)}{\cancel{2}} + n$$

$$= \frac{2n(n+1)(2n+1)}{3} - \frac{6n(n+1)}{3} + \frac{3n}{3}$$

$$= \frac{n}{3}\left[2(n+1)(2n+1) - 6(n+1) + 3\right]$$

$$= \frac{n}{3}\left[4n^2 + 6n + 2 - 6n - 6 + 3\right]$$

$$= \frac{1}{3}n(4n^2-1), \text{ as required.}$$

$$\sum_{r=1}^{n}1 = \underbrace{1+1+1+\ \ldots\ +1}_{n \text{ times}} = n$$

It is a common error to write $\sum_{r=1}^{n}1 = 1$.

After "cancelling" the fractions, you should put all terms over a common denominator, here 3.

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Review Exercise
Exercise A, Question 24

Question:

Use standard formulae to show that $\sum_{r=1}^{n} r(r^2-3) = \frac{1}{4}n(n+1)(n-2)(n+3)$.

Solution:
img src=

$$\sum_{r=1}^{n} r(r^2-3) = \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r$$
$$= \frac{n^2(n+1)^2}{4} - \frac{3n(n+1)}{2}$$
$$= \frac{n^2(n+1)^2}{4} - \frac{6n(n+1)}{4}$$
$$= \frac{n(n+1)}{4}[n(n+1)-6]$$
$$= \frac{n(n+1)}{4}[n^2+n-6]$$
$$= \frac{1}{4}n(n+1)(n-2)(n+3), \text{ as required.}$$

After putting both terms over a common denominator, look for the common factors of the terms, here shown in **bold** ;
$$\frac{\boldsymbol{n}^2(\boldsymbol{n+1})^2}{4} - \frac{6\boldsymbol{n}(\boldsymbol{n+1})}{4}.$$
You take these, together with the common denominator 4, outside a bracket;
$$\frac{\boldsymbol{n}(\boldsymbol{n+1})}{4}[n(n+1)-6].$$
You need to be careful with the squared terms.

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Review Exercise
Exercise A, Question 25

Question:

a Use standard formulae to show that $\sum_{r=1}^{n} r(2r-1) = \frac{n(n+1)(4n-1)}{6}$.

b Hence, evaluate $\sum_{r=11}^{30} r(2r-1)$.

Solution:

a

$$\sum_{r=1}^{n} r(2r-1) = \sum_{r=1}^{n}(2r^2 - r)$$

$$= 2\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r$$

$$= \frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

$$= \frac{2n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{6}$$

You put the expressions over a common denominator, here 6, and then look for the common factors of the expressions, here n and $(n+1)$.

$$= \frac{n(n+1)}{6}[2(2n+1)-3]$$

$$= \frac{n(n+1)}{6}[4n+2-3]$$

$$= \frac{n(n+1)(4n-1)}{6}, \text{ as required.}$$

b

$$\sum_{r=11}^{30} r(2r-1) = \sum_{r=1}^{30} r(2r-1) - \sum_{r=1}^{10} r(2r-1)$$

$\sum_{r=11}^{30} f(r) = \sum_{r=1}^{30} f(r) - \sum_{r=1}^{10} f(r)$.

You find the sum from the 11th to the 30th term by subtracting the sum from the first to the 10th term from the sum from the first to the 30th term. It is a common error to subtract one term too many, in this case the 11th term. The sum you are finding starts with the 11th term. You must not subtract it from the series – you have to leave it in the series.

Substituting $n = 30$ and $n = 10$ into the result in part (a).

$$\sum_{r=11}^{30} r(2r-1) = \frac{30\times31\times119}{6} - \frac{10\times11\times39}{6}$$

$$= 18\,445 - 715$$

$$= 17\,730$$

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Review Exercise
Exercise A, Question 26

Question:

Evaluate $\sum_{r=0}^{12}(r^2+2^r)$.

Solution:

$$\sum_{r=0}^{12}\left(r^2+2^r\right)=\sum_{r=0}^{12}r^2+\sum_{r=0}^{12}2^r$$

This question asks you to carry out two different sums. The first involves $\sum_{r=1}^{n}r^2$, which you learnt in Chapter 5 of this book. The other is a Geometric Series which you can find in Chapter 7 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 2.

$$\sum_{r=0}^{12}r^2=\sum_{r=1}^{12}r^2$$

$$\sum_{r=1}^{n}r^2=\frac{n(n+1)(2n+1)}{6}$$

Substituting $n=12$ into the standard formula.

The first term in this series, corresponding to $r=0$ is 0^2. This obviously does not add anything to the series, so you can start the summation from 1 and use the standard formula with $n=12$.

$$\sum_{r=1}^{12}r^2=\frac{12(12+1)(2\times12+1)}{6}=\frac{12\times13\times25}{6}$$

$$=650$$

$$\sum_{r=0}^{12}2^r=2^0+2^1+2^2+\ \dots+2^{12}$$

This is a Geometric Series with $a=2^0=1$, $r=2$ and $n=13$.

The first term in the Geometric Series is $2^0=1$ and you must include this in the sum. With this, there are 13 terms in the series.

Using the formula $S=\dfrac{a\left(r^n-1\right)}{r-1}$,

$$\sum_{r=0}^{12}2^r=\frac{1\left(2^{13}-1\right)}{2-1}=2^{13}-1=8191$$

Combining the two results

$$\sum_{r=0}^{12}\left(r^2+2^r\right)=650+8191=8841$$

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Review Exercise
Exercise A, Question 27

Question:

Evaluate $\sum_{r=1}^{50}(r+1)(r+2)$.

Solution:

$$\sum_{r=1}^{n}(r+1)(r+2)=\sum_{r=1}^{n}r^2+3\sum_{r=1}^{n}r+\sum_{r=1}^{n}2$$

You have not been asked to show that any particular formula in n is true but you have to get an expression for the summation in terms of n and then substitute $n=50$ into it.

$$=\frac{n(n+1)(2n+1)}{6}+\frac{3n(n+1)}{2}+2n$$

As you have not been asked to show that any formula is true, you need not look for any common factors in these terms. You can use the whole expression as it is.

Substituting $n=50$

$$\sum_{r=1}^{50}(r+1)(r+2)=\frac{50\times 51\times 101}{6}+\frac{3\times 50\times 51}{2}+2\times 50$$

$$=42\,925+3825+100=46\,850$$

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Review Exercise
Exercise A, Question 28

Question:

Use standard formulae to show that $\sum_{r=1}^{n} r(r^2-n) = \frac{n^2(n^2-1)}{4}$.

Solution:

$$\sum_{r=1}^{n} r(r^2-n) = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} nr$$

$$= \sum_{r=1}^{n} r^3 - n\sum_{r=1}^{n} r$$

$$= \frac{n^2(n+1)^2}{4} - n\times\frac{n(n+1)}{2}$$

$$= \frac{n^2(n+1)^2}{4} - \frac{2n^2(n+1)}{4}$$

$$= \frac{n^2(n+1)}{4}\left[(n+1)-2\right]$$

$$= \frac{n^2(n+1)(n-1)}{4}$$

$$= \frac{n^2(n^2-1)}{4}, \text{ as required.}$$

In $\sum_{r=1}^{n} nr$, the r ranges from 1 to n but the n does not change; n is a constant. So

$$\sum_{r=1}^{n} nr = n\times1 + n\times2 + n\times3 + \dots + n\times n$$

$$= n\times(1+2+3+\dots+n) = n\times\frac{n(n+1)}{2}$$

After putting both terms over a common denominator, look for the common factors of the terms, here shown in **bold**;

$$\frac{\boldsymbol{n}^2(\boldsymbol{n}+\mathbf{1})^2}{\mathbf{4}} - \frac{2\boldsymbol{n}^2(\boldsymbol{n}+\mathbf{1})}{\mathbf{4}}.$$

You take these, together with the common denominator 4, outside a bracket;

$$\frac{\boldsymbol{n}^2(\boldsymbol{n}+\mathbf{1})}{\mathbf{4}}\left[(n+1)-2\right].$$

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Review Exercise
Exercise A, Question 29

Question:

a Use standard formulae to show that $\sum_{r=1}^{n} r(3r+1) = n(n+1)^2$.

b Hence evaluate $\sum_{r=40}^{100} r(3r+1)$.

Solution:

a $\sum_{r=1}^{n} r(3r+1) = 3\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$= \frac{\cancel{3}^1 n(n+1)(2n+1)}{\cancel{6}^2} + \frac{n(n+1)}{2}$

$= \frac{n(n+1)}{2}\left[(2n+1)+1\right]$

$= \frac{n(n+1)(2n+2)}{2} = \frac{n(n+1)\cancel{2}(n+1)}{\cancel{2}}$

$= n(n+1)^2$, as required.

b $\sum_{r=40}^{100} r(3r+1) = \sum_{r=1}^{100} r(3r+1) - \sum_{r=1}^{39} r(3r+1)$

Substituting $n = 100$ and $n = 39$ into the result in part (a).

$\sum_{r=40}^{100} r(3r+1) = 100 \times 101^2 - 39 \times 40^2$

$= 1\,020\,100 - 62\,400$

$= 957\,700$

$\sum_{r=40}^{100} \mathrm{f}(r) = \sum_{r=1}^{100} \mathrm{f}(r) - \sum_{r=1}^{39} \mathrm{f}(r)$.

You find the sum from the 40th to the 100th term by subtracting the sum from the first to the 39th term from the sum from the first to the 100th term.

It is a common error to subtract one term too many, in this case the 40th term. The sum you are finding starts with the 40th term. You must not subtract the 40th term from the series – you have to leave it in the series.

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Review Exercise
Exercise A, Question 30

Question:

a Show that $\sum_{r=1}^{n}(2r-1)(2r+3) = \frac{n}{3}(4n^2+12n-1)$.

b Hence find $\sum_{r=5}^{35}(2r-1)(2r+3)$.

Solution:

a
$$\sum_{r=1}^{n}(2r-1)(2r+3) = \sum_{r=1}^{n}(4r^2+4r-3)$$
$$= 4\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - \sum_{r=1}^{n} 3$$
$$= \frac{\cancel{4}^2 n(n+1)(2n+1)}{\cancel{6}^3} + \frac{\cancel{4}^2 n(n+1)}{\cancel{2}} - 3n$$
$$= \frac{2n(n+1)(2n+1)}{3} + \frac{6n(n+1)}{3} - \frac{9n}{3}$$
$$= \frac{n}{3}\left[2(n+1)(2n+1)+6(n+1)-9n\right]$$
$$= \frac{n}{3}\left[4n^2+6n+2+6n+6-9\right]$$
$$= \frac{n}{3}(4n^2+12n-1) \text{, as required.}$$

$\sum_{r=1}^{n} 3 = 3+3+3+ \dots +3 = 3n$

It is a common error to write $\sum_{r=1}^{n} 3 = 3$.

After "cancelling" fractions, put all of the expressions over a common denominator, here 3.
You then look for any factors common to all three expressions.
Here there is only one, n.

b
$$\sum_{r=5}^{35}(2r-1)(2r+3)$$
$$= \sum_{r=1}^{35}(2r-1)(2r+3) - \sum_{r=1}^{4}(2r-1)(2r+3)$$

You find the sum from the 5th to the 35th term by subtracting the sum from the first to the 4th term from the sum from the first to the 35th term.

Substituting $n=35$ and $n=4$ into the result in part (a)
$$\sum_{r=5}^{35}(2r-1)(2r+3)$$
$$= \frac{35}{3}(4\times35^2+12\times35-1) - \frac{4}{3}(4\times4^2+12\times4-1)$$
$$= 62\,055 - 148 = 61\,907$$

You use the expression you have proved in part (a) to complete the question.

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Review Exercise
Exercise A, Question 31

Question:

a Use standard formulae to show that $\sum_{r=1}^{n}(6r^2+4r-5)=n(2n^2+5n-2)$.

b Hence calculate the value of $\sum_{r=10}^{25}(6r^2+4r-5)$.

Solution:

a

$$\sum_{r=1}^{n}\left(6r^2+4r-5\right)=6\sum_{r=1}^{n}r^2+4\sum_{r=1}^{n}r-\sum_{r=1}^{n}5$$

$$=\frac{\cancel{6}n(n+1)(2n+1)}{\cancel{6}}+\frac{\cancel{4}^2n(n+1)}{\cancel{2}}-5n$$

$$=n(n+1)(2n+1)+2n(n+1)-5n$$

$$=n\left[(n+1)(2n+1)+2(n+1)-5\right]$$

$$=n\left[2n^2+3n+1+2n+2-5\right]$$

$$=n\left(2n^2+5n-2\right)\text{, as required.}$$

A common error with the last term is to write $-\sum_{r=1}^{n}5=-5$. Correctly:

$$-\sum_{r=1}^{n}5=-(5+5+5+\ \dots\ +5)$$

$$=-5(\underbrace{1+1+1+\ \dots\ +1}_{n\text{ times}})$$

$$=-5n$$

b

$$\sum_{r=10}^{25}\left(6r^2+4r-5\right)=\sum_{r=1}^{25}\left(6r^2+4r-5\right)-\sum_{r=1}^{9}\left(6r^2+4r-5\right)$$

Substituting $n=25$ and $n=9$ into the result in part (a)

$$\sum_{r=10}^{25}\left(6r^2+4r-5\right)$$

$$=25\left(2\times25^2+5\times25-2\right)-9\left(2\times9^2+5\times9-2\right)$$

$$=34\,325-1845=32\,480$$

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Review Exercise
Exercise A, Question 32

Question:

a Use standard formulae to show that $\sum_{r=1}^{n}(r+1)(r+5)=\frac{1}{6}n(n+7)(2n+7)$.

b Hence calculate the value of $\sum_{r=10}^{40}(r+1)(r+5)$.

Solution:

a $\sum_{r=1}^{n}(r+1)(r+5)=\sum_{r=1}^{n}(r^2+6r+5)$

$$=\sum_{r=1}^{n}r^2+6\sum_{r=1}^{n}r+\sum_{r=1}^{n}5$$

$$=\frac{n(n+1)(2n+1)}{6}+\frac{6n(n+1)}{2}+5n$$

$$=\frac{n(n+1)(2n+1)}{6}+\frac{18n(n+1)}{6}+\frac{30n}{6}$$

$$=\frac{n}{6}\left[(n+1)(2n+1)+18(n+1)+30\right]$$

$$=\frac{n}{6}\left[2n^2+3n+1+18n+18+30\right]$$

$$=\frac{n}{6}\left(n^2+21n+49\right)$$

$$=\frac{1}{5}n(n+7)(2n+7)\text{, as required.}$$

As the question prints the answer, factorising the quadratic expression gives no difficulty, but you should check your solution by multiplying out the brackets in the answer. This helps you to correct any errors that you may have made in your working. In this case, the check is

$$(n+7)(2n+7)=2n^2+7n+14n+49$$
$$=2n^2+21n+49.$$

This checks and you can be confident the working is correct.

b $\sum_{r=10}^{40}(r+1)(r+5)=\sum_{r=1}^{40}(r+1)(r+5)-\sum_{r=1}^{9}(r+1)(r+5)$

Substituting $n=40$ and $n=9$ into the result in part (a)

$$\sum_{r=10}^{40}(r+1)(r+5)=\frac{1}{6}\times40\times47\times87-\frac{1}{6}\times9\times16\times25$$

$$=27\,260-600=26\,660$$

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Review Exercise
Exercise A, Question 33

Question:

a Use standard formulae to show that $\sum_{r=1}^{n} r^2(r+1) = \frac{n(n+1)(3n^2+7n+2)}{12}$.

b Find $\sum_{r=4}^{30} (2r)^2(2r+2)$.

Solution:

a $\sum_{r=1}^{n} r^2(r+1) = \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2$

$= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}$

$= \frac{3n^2(n+1)^2}{12} + \frac{2n(n+1)(2n+1)}{12}$

$= \frac{n(n+1)}{12}[3n(n+1)+2(2n+1)]$

After putting both terms over the common denominator 12, find the common factors of the terms, here shown in **bold**;

$\frac{3\boldsymbol{n}^2(\boldsymbol{n+1})^2}{12} + \frac{2\boldsymbol{n}(\boldsymbol{n+1})(2n+1)}{12}$

$= \frac{\boldsymbol{n}(\boldsymbol{n+1})}{12}[3n(n+1)+2(2n+1)]$.

$= \frac{n(n+1)}{12}[3n^2+3n+4n+2]$

$= \frac{n(n+1)(3n^2+7n+2)}{12}$, as required.

b $(2r)^2(2r+2) = 4r^2 \times 2(r+1) = 8r^2(r+1)$

$\sum_{r=4}^{30} (2r)^2(2r+2) = 8\sum_{r=4}^{30} r^2(r+1)$

Each term, $(2r)^2(2r+2)$, in the summation in part (b) is eight times the corresponding term, $r^2(r+1)$, in part (a). The key idea is then to find $\sum_{r=4}^{30} r^2(r+1)$ and multiply this by 8.

$= 8\sum_{r=4}^{30} r^2(r+1) = 8\left(\sum_{r=1}^{30} r^2(r+1) - \sum_{r=1}^{3} r^2(r+1)\right)$

Substituting $n=30$ and $n=3$ into the result in part (a)

$\sum_{r=4}^{30} (2r)^2(2r+2)$

$= 8\left(\frac{30\times31\times(3\times30^2+7\times30+2)}{12} - \frac{3\times4\times(3\times3^2+7\times3+2)}{12}\right)$

$= 8(225\,680 - 50) = 8 \times 225\,630 = 1\,805\,040$

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Review Exercise
Exercise A, Question 34

Question:

Using the formula $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$,

a show that $\sum_{r=1}^{n}(4r^2 - 1) = \frac{n}{3}(4n^2 + 6n - 1)$.

Given that $\sum_{r=1}^{12}(4r^2 + kr - 1) = 2120$, where k is a constant,

b find the value of k.

Solution:

a $\sum_{r=1}^{n}(4r^2-1) = 4\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} 1$

$= \frac{\cancel{4}^2 n(n+1)(2n+1)}{\cancel{6}^3} - n$

$= \frac{2n(n+1)(2n+1)}{3} - \frac{3n}{3}$

$= \frac{n}{3}\left[2(n+1)(2n+1) - 3\right]$

$= \frac{n}{3}\left[4n^2 + 6n + 2 - 3\right]$

$= \frac{n}{3}(4n^2 + 6n - 1)$, as required.

$\sum_{r=1}^{n} 1 = 1 + 1 + 1 + \ldots + 1 = n$

It is a common error to write $\sum_{r=1}^{n} 1 = 1$.

After "cancelling" fractions, put all of the expressions over a common denominator, here 3.
You then look for any factors common to all three expressions. Here there is only one, n.

b $\sum_{r=1}^{12}(4r^2 + kr - 1) = 2120$

$\sum_{r=1}^{12}(4r^2 - 1) + k\sum_{r=1}^{12} r = 2120 \quad \ldots *$

Using the result in part (a) with $n = 12$

$\sum_{r=1}^{12}(4r^2-1) = \frac{12}{3}(4 \times 12^2 + 6 \times 12 - 1)$

$= 2588$

Using the standard result $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

with $n = 12$

$\sum_{r=1}^{12} r = \frac{12(12+1)}{2} = 78$

Equation $*$ now becomes

$2588 + 78k = 2120$

$k = \frac{2120 - 2588}{78} = -6$

$(4r^2 + kr - 1)$ can be written as $(4r^2 - 1) + kr$.

You can then evaluate $\sum_{r=1}^{12}(4r^2 - 1)$, using part (a),

and $\sum_{r=1}^{12} r$, using a formula you learnt in Chapter 5.

The relation in the question then becomes an equation in k which you can solve.

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Review Exercise
Exercise A, Question 35

Question:

a Use standard formulae to show that $\sum_{r=1}^{n} r(3r-5) = n(n+1)(n-2)$.

b Hence show that $\sum_{r=n}^{2n} r(3r-5) = 7n(n^2-1)$.

Solution:

a $\sum_{r=1}^{n} r(3r-5) = 3\sum_{r=1}^{n} r^2 - 5\sum_{r=1}^{n} r$

$= \frac{\cancel{3}^1 n(n+1)(2n+1)}{\cancel{6}^2} - \frac{5n(n+1)}{2}$

$= \frac{n(n+1)}{2}[2n+1-5]$

$= \frac{n(n+1)\cancel{2}(n-2)}{\cancel{2}}$

$= n(n+1)(n-2)$, as required.

Look for the common factors of the terms, here shown in **bold** ;

$\frac{\boldsymbol{n(n+1)}(2n+1)}{2} - \frac{5\boldsymbol{n(n+1)}}{2}$.

Take the common factors, together with the common denominator 2, outside a bracket;

$\frac{\boldsymbol{n(n+1)}}{2}[(2n+1)-5]$.

b $\sum_{r=n}^{2n} r(3r-5) = \sum_{r=1}^{2n} r(3r-5) - \sum_{r=1}^{n-1} r(3r-5)$

Using the result in part (a), replacing n by $2n$ and $n-1$.

$\sum_{r=n}^{2n} r(3r-5) = 2n(2n+1)(2n-2) - (n-1)n(n-3)$

$= 4n(2n+1)(n-1) - (n-1)n(n-3)$

$= n(n-1)[4(2n+1) - (n-3)]$

$= n(n-1)[8n+4-n+3]$

$= n(n-1)(7n+7)$

$= 7n(n-1)(n+1)$

$= 7n(n^2-1)$, as required.

$\sum_{r=n}^{2n} r(3r-5) = \sum_{r=1}^{2n} \mathrm{f}(r) - \sum_{r=1}^{n-1} \mathrm{f}(r)$

To find an expression for $\sum_{r=1}^{2n} \mathrm{f}(r)$, you replace the n in the result in part (a) by $2n$;

$n(n+1)(n-2)$

becomes $2n(2n+1)(2n-2)$.

To find an expression for $\sum_{r=1}^{n-1} \mathrm{f}(r)$, you replace the n in the result in part (a) by $n-1$;

$n(n+1)(n-2)$

becomes $(n-1)((n-1)+1)((n-1)-2)$

$= (n-1)n(n-3)$.

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Review Exercise
Exercise A, Question 36

Question:

a Use standard formulae to show that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$.

b Hence, or otherwise, show that $\sum_{r=n}^{3n} r(r+1) = \frac{1}{3}n(2n+1)(pn+q)$, stating the values of the integers p and q.

Solution:

a $\sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{6}$

After putting the expressions over a common denominator 6, you look for any factors common to both expressions. Here there are two, n and $(n+1)$.

$= \frac{n(n+1)}{6}[2n+1+3]$

$= \frac{n(n+1)\cancel{2}^{1}(n+2)}{\cancel{6}^{3}}$

$= \frac{1}{3}n(n+1)(n+2)$, as required.

b $\sum_{r=n}^{3n} r(r+1) = \sum_{r=1}^{3n} r(r+1) - \sum_{r=1}^{n-1} r(r+1)$

To find an expression for $\sum_{r=1}^{n-1} r(r+1)$, you replace the n in the result in part (a) by $n-1$;

$\frac{1}{3}n(n+1)(n+2)$

becomes $\frac{1}{3}(n-1)((n-1)+1)((n-1)+2)$

$= \frac{1}{3}(n-1)n(n+1)$.

$= \frac{1}{3}3n(3n+1)(3n+2) - \frac{1}{3}(n-1)n(n+1)$

$= \frac{1}{3}n[3(3n+1)(3n+2) - (n-1)(n+1)]$

$= \frac{1}{3}n[27n^2 + 27n + 6 - (n^2 - 1)]$

$= \frac{1}{3}n(26n^2 + 27n + 7)$

$= \frac{1}{3}n(2n+1)(13n+7)$

$p = 13, q = 7$

As you are given that $(2n+1)$ is one factor of $26n^2 + 27n + 7$, the other can just be written down. $(2n+1)(pn+q) = 26n^2 + 27n + 7$, only if $2p = 26$ and $1q = 7$

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Review Exercise
Exercise A, Question 37

Question:

Given that $\sum_{r=1}^{n} r^2(r-1) = \frac{1}{12}n(n+1)(pn^2 + qn + r)$,

a find the values of p, q and r.

b Hence evaluate $\sum_{r=50}^{100} r^2(r-1)$.

Solution:

a $\sum_{r=1}^{n} r^2(r-1) = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^2$

$$= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{3n^2(n+1)^2}{12} - \frac{2n(n+1)(2n+1)}{12}$$

$$= \frac{n(n+1)}{12}\left[3n(n+1) - 2(2n+1)\right]$$

$$= \frac{n(n+1)}{12}\left[3n^2 + 3n - 4n - 2\right]$$

$$= \frac{1}{12}n(n+1)(3n^2 - n - 2)$$

$p = 3,\ q = -1,\ r = -2$

After putting the expressions over a common denominator 12, you look for any factors common to both expressions. Here there are two, n and $(n+1)$.

b $\sum_{r=50}^{100} r^2(r-1) = \sum_{r=1}^{100} r^2(r-1) - \sum_{r=1}^{49} r^2(r-1)$

$$= \frac{1}{12} \times 100 \times 101 \times (3 \times 100^2 - 100 - 2)$$

$$- \frac{1}{12} \times 49 \times 50 \times (3 \times 49^2 - 49 - 2)$$

$$= 25\,164\,150 - 1\,460\,200$$

$$= 23\,703\,950$$

$\sum_{r=50}^{100} f(r) = \sum_{r=1}^{100} f(r) - \sum_{r=1}^{49} f(r)$.
You find the sum from the 50th to the 100th term by subtracting the sum from the first to the 49th term from the sum from the first to the 100th term.
It is a common error to subtract one term too many, in this case the 50th term. The sum you are finding starts with the 50th term. You must not remove it from the series.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 38

Question:

a Use standard formula to show that $\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7)$.

b Hence, or otherwise, find the value of $\sum_{r=1}^{10} (r+2)\log_4 2^r$.

Solution:

a $\sum_{r=1}^{n} r(r+2) = \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$

$= \frac{n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$

$= \frac{n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{6}$

$= \frac{n(n+1)}{6}[2n+1+6]$

$= \frac{1}{6}n(n+1)(2n+7)$, as required.

In part (b), you need to use the properties of logarithms you learnt in the C2 course. You can find this material in Chapter 3 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 2.

b $\sum_{r=1}^{10} (r+2)\log_4 2^r = \sum_{r=1}^{10} (r+2) r \log_4 2$

$= \log_4 2 \sum_{r=1}^{10} r(r+2)$

$= \log_4 4^{\frac{1}{2}} \sum_{r=1}^{10} r(r+2)$

The power law of logarithms, $\log_a x^k = k\log_a x$, is used twice.

$= \frac{1}{2}\log_4 4 \sum_{r=1}^{10} r(r+2)$

$\log_a a = 1$, for any positive a.

$= \frac{1}{2}\sum_{r=1}^{10} r(r+2)$

$= \frac{1}{2} \times \frac{1}{6} \times 10 \times 11 \times 27$

$= 247\frac{1}{2}$

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 39

Question:

Use the method of mathematical induction to prove that, for all positive integers n, $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$.

Solution:

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

All inductions need to be shown to be true for a small number, usually 1.

Let $n = 1$.

The left-hand side becomes

$$\sum_{r=1}^{1} \frac{1}{r(r+1)} = \frac{1}{1 \times 2} = \frac{1}{2}$$

$\sum_{r=1}^{1} \frac{1}{r(r+1)}$ consists of just one term. That is $\frac{1}{r(r+1)}$ with 1 substituted for r.

The right-hand side becomes

$$\frac{1}{1+1} = \frac{1}{2}$$

The left-hand side and the right-hand side are equal and so the summation is true for $n = 1$.

Assume the summation is true for $n = k$.

That is $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$ … … $*$

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$$

The sum from 1 to $k+1$ is the sum from 1 to k plus one extra term. In this case, the extra term is found by replacing each r in $\frac{1}{r(r+1)}$ by $k+1$.

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}, \text{ using } *$$

$$= \frac{k(k+1)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1}$$

Keep in mind what you are aiming for as you work out the algebra. You are looking to prove that the summation is true for $n = k+1$, so you are trying to reach $\frac{n}{n+1}$ with the n replaced by $k+1$.

This is the result obtained by substituting $n = k+1$ into the right-hand side of the summation and so the summation is true for $n = k+1$.

The summation is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the summation is true for all positive integers n.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 40

Question:

Use the method of mathematical induction to prove that $\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$.

Solution:

$$\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$$

Let $n=1$.
The left-hand side becomes

$$\sum_{r=1}^{1} r(r+3) = 1(1+3) = 4$$

$\sum_{r=1}^{1} r(r+3)$ consists of just one term. That is $r(r+3)$ with 1 substituted for r.

The right-hand side becomes

$$\frac{1}{3}\times 1(1+1)(1+5) = \frac{1}{3}\times 2\times 6 = 4$$

The left-hand side and the right-hand side are equal and so the summation is true for $n=1$.

Assume the summation is true for $n=k$.

That is $\sum_{r=1}^{k} r(r+3) = \frac{1}{3}k(k+1)(k+5)$ … … ∗

This is often called the **induction hypothesis**.

$$\sum_{r=1}^{k+1} r(r+3) = \sum_{r=1}^{k} r(r+3) + (k+1)(k+4)$$

The sum from 1 to $k+1$ is the sum from 1 to k plus one extra term.
In this case, the extra term is found by replacing each r in $r(r+3)$ by $k+1$.

$$= \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4)\text{, using } \ast$$

$$= \frac{1}{3}k(k+1)(k+5) + \frac{3}{3}(k+1)(k+4)$$

$$= \frac{1}{3}(k+1)\left[k(k+5)+3(k+4)\right]$$

$$= \frac{1}{3}(k+1)\left[k^2+5k+3k+12\right]$$

$$= \frac{1}{3}(k+1)\left[k^2+8k+12\right]$$

$$= \frac{1}{3}(k+1)(k+2)(k+6)$$

$$= \frac{1}{3}(k+1)((k+1)+1)((k+1)+5)$$

Multiplying out the brackets would give you an awkward cubic expression which would be difficult to factorise. You should try to simplify the working by looking for any common factors and taking them outside a bracket. Here $(k+1)$ is a common factor.

This expression is $\frac{1}{3}n(n+1)(n+5)$ with each n replaced by $k+1$.

This is the result obtained by substituting $n=k+1$ into the right-hand side of the summation and so the summation is true for $n=k+1$.

The summation is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the summation is true for all positive integers n.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 41

Question:

Prove by induction that, for $n \varepsilon \mathbb{Z}^{+}$, $\sum_{r=1}^{n} (2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$.

Solution:

$$\sum_{r=1}^{n}(2r-1)^2=\frac{1}{3}n(2n-1)(2n+1)$$

Let $n=1$.

The left-hand side becomes

$$\sum_{r=1}^{1}(2r-1)^2=(2-1)^2=1^2=1$$

$\sum_{r=1}^{1}(2r-1)^2$ consists of just one term. That is $(2r-1)^2$ with 1 substituted for r.

The right-hand side becomes

$$\frac{1}{3}\times 1(2-1)(2+1)=\frac{1}{3}\times 1\times 1\times 3=1$$

The left-hand side and the right-hand side are equal and so the summation is true for $n=1$.

Assume the summation is true for $n=k$.

That is $\sum_{r=1}^{k}(2r-1)^2=\frac{1}{3}k(2k-1)(2k+1)$ … … $*$

$$\sum_{r=1}^{k+1}(2r-1)^2=\sum_{r=1}^{k}(2r-1)^2+(2k+1)^2$$

The sum from 1 to $k+1$ is the sum from 1 to k plus one extra term. In this case, the extra term is found by replacing the r in $(2r-1)^2$ by $k+1$. Giving $(2(k+1)-1)^2=(2k+2-1)^2=(2k+1)^2$

$$=\frac{1}{3}k(2k-1)(2k+1)+\frac{3}{3}(2k+1)^2\text{, using } *$$

$$=\frac{1}{3}(2k+1)\left[k(2k-1)+3(2k+1)\right]$$

Multiplying out the brackets would give you an awkward cubic expression which would be difficult to factorise. Look for any common factors and take them outside a bracket. Here $(2k+1)$ is a common factor.

$$=\frac{1}{3}(2k+1)\left[2k^2+5k+3\right]$$

$$=\frac{1}{3}(2k+1)(k+1)(2k+3)$$

$$=\frac{1}{3}(k+1)(2(k+1)-1)(2(k+1)+1)$$

This expression is $\frac{1}{3}n(2n-1)(2n+1)$ with each n replaced by $k+1$.

This is the result obtained by substituting $n=k+1$ into the right-hand side of the summation and so the summation is true for $n=k+1$.

The summation is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the summation is true for all positive integers n.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 42

Question:

The rth term a_r in a series is given by $a_r = r(r+1)(2r+1)$.

Prove, by mathematial induction, that the sum of the first n terms of the series is $\frac{1}{2}n(n+1)^2(n+2)$.

Solution:

$$\sum_{r=1}^{n} a_r = \sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2)$$

Let $n=1$.

All inductions need to be shown to be true for a small number, usually 1.

The left-hand side becomes

$$\sum_{r=1}^{1} r(r+1)(2r+1) = 1\times2\times3 = 6$$

The right-hand side becomes

$$\frac{1}{2}\times1\times2^2\times3 = 6$$

The left-hand side and the right-hand side are equal and so the summation is true for $n=1$.

Assume the summation is true for $n=k$.

That is $\sum_{r=1}^{k} r(r+1)(2r+1) = \frac{1}{2}k(k+1)^2(k+2)$ ∗

$$\sum_{r=1}^{k+1} r(r+1)(2r+1) = \sum_{r=1}^{k} r(r+1)(2r+1) + (k+1)(k+2)(2k+3)$$

$= \frac{1}{2}k(k+1)^2(k+2) + \frac{2}{2}(k+1)(k+2)(2k+3)$, using ∗

$= \frac{1}{2}(k+1)(k+2)\left[k(k+1)+2(2k+3)\right]$

Fractions need to be expressed to the same denominator before factorising. The form of the answer shows that you need to have $\frac{1}{2}$ as a common factor and it helps you to write $\frac{2}{2}$ before the second term on the right-hand side of the summation.

$= \frac{1}{2}(k+1)(k+2)\left[k^2+5k+6\right]$

$= \frac{1}{2}(k+1)(k+2)(k+2)(k+3)$

$= \frac{1}{2}(k+1)(k+2)^2(k+3)$

$= \frac{1}{2}(k+1)((k+1)+1)^2((k+1)+2)$

This expression is $\frac{1}{2}n(n+1)^2(2n+1)$ with each n replaced by $k+1$.

This is the result obtained by substituting $n=k+1$ into the right-hand side of the summation and so the summation is true for $n=k+1$.

The summation is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the summation is true for all positive integers n.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 43

Question:

Prove, by induction, that $\sum_{r=1}^{n} r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2)$.

Solution:

$$\sum_{r=1}^{n} r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2)$$

Let $n=1$.

The left-hand side becomes

$$\sum_{r=1}^{1} r^2(r-1) = 1^2 \times (1-1) = 0$$

$\sum_{r=1}^{1} r^2(r-1)$ consists of just one term. That is $r^2(r-1)$ with 1 substituted for r. In this case, because of the bracket, this clearly gives 0.

The right-hand side becomes

$$\frac{1}{12} \times 1 \times (1-1) \times (1+1) \times (3+2)$$

$$= \frac{1}{12} \times 1 \times 0 \times 2 \times 5 = 0$$

The left-hand side and the right-hand side are equal and so the summation is true for $n=1$.

Assume the summation is true for $n=k$.

That is $\sum_{r=1}^{k} r^2(r-1) = \frac{1}{12}k(k-1)(k+1)(3k+2)$ ... ... $*$

$$\sum_{r=1}^{k+1} r^2(r-1) = \sum_{r=1}^{k} r^2(r-1) + (k+1)^2(k+1-1)$$

$$= \frac{1}{12}k(k-1)(k+1)(3k+2) + \frac{12}{12}k(k+1)^2, \text{ using } *$$

The common factors in these two terms are $\frac{1}{12}$, k and $(k+1)$.

$$= \frac{1}{12}k(k+1)\left[(k-1)(3k+2) + 12(k+1)\right]$$

$$= \frac{1}{12}k(k+1)\left[3k^2 - k - 2 + 12k + 12\right]$$

$$= \frac{1}{12}k(k+1)\left[3k^2 + 11k + 10\right]$$

$$= \frac{1}{12}k(k+1)(k+2)(3k+5)$$

$$= \frac{1}{12}(k+1)((k+1)-1)((k+1)+1)(3(k+1)+2)$$

Rearrange this expression so that it is the right-hand side of the summation with n replaced by $k+1$.

This is the result obtained by substituting $n=k+1$ into the right-hand side of the summation and so the summation is true for $n=k+1$.

The summation is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the summation is true for all positive integers n.

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Review Exercise
Exercise A, Question 44

Question:

Given that $u_1 = 8$ and $u_{n+1} = 4u_n - 9n$, use mathematical induction to prove that $u_n = 4^n + 3n + 1$, $n \varepsilon \mathbb{Z}^+$.

Solution:

$u_n = 4^n + 3n + 1$

Let $n = 1$

$u_1 = 4^1 + 3 \times 1 + 1 = 4 + 3 + 1 = 8$

As the question gives $u_1 = 8$, the formula is true for $n = 1$.

All inductions need to be shown to be true for a small number, usually 1. In this question $u_1 = 8$ is part of the data of the question and you have to start by showing that $u_n = 4^n + 3n + 1$ satisfies $u_1 = 8$.

Assume the formula is true for $n = k$.

That is $u_k = 4^k + 3k + 1$ ... ... $\ast$

The **induction hypothesis** is just the formula you are asked to prove with the ns replaced by ks.

$u_{k+1} = 4u_k - 9k$

$= 4(4^k + 3k + 1) - 9k$, using $\ast$

$= 4^{k+1} + 12k + 4 - 9k$

$= 4^{k+1} + 3k + 4$

$= 4^{k+1} + 3(k+1) + 1$

The induction hypothesis allows you to substitute $4^k + 3k + 1$ for u_k.

This is the result obtained by substituting $n = k + 1$ into the formula $u_n = 4^n + 3n + 1$ and so the formula is true for $n = k + 1$.

The formula is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k + 1$.

By mathematical induction the formula is true for all positive integers n.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 45

Question:

Given that $u_1 = 0$ and $u_{r+1} = 2r - u_r$, use mathematical induction to prove that $2u_n = 2n - 1 + (-1)^n$, $n \varepsilon \mathbb{Z}^+$.

Solution:

$2u_n = 2n - 1 + (-1)^n$

Let $n = 1$

$2u_1 = 2 - 1 + (-1)^1 = 2 - 1 - 1 = 0 \Rightarrow u_1 = 0$

As the question gives $u_1 = 0$, the formula is true for $n = 1$.

Assume the formula is true for $n = k$.

That is $2u_k = 2k - 1 + (-1)^k$ … … ∗

$u_{k+1} = 2k - u_k$

Replacing the r by a k in $u_{r+1} = 2r - u_r$.
This question has used r in the data in the question where n has been used in the previous questions in this exercise. The letters used are symbols and which particular letter is used is makes no difference to the question or the way you solve it .

$2u_{k+1} = 4k - 2u_k = 4k - \left(2k - 1 + (-1)^k\right)$, using ∗

$= 4k - 2k + 1 - (-1)^k$

$= 2k + 1 + (-1)^{k+1}$

$-(-1)^k = (-1)(-1)^k = (-1)^1(-1)^k = (-1)^{k+1}$

$= 2(k+1) - 1 + (-1)^{k+1}$

This is the result obtained by substituting $n = k + 1$ into the formula $2u_n = 2n - 1 + (-1)^n$ and so the formula is true for $n = k + 1$.

The formula is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k + 1$.

By mathematical induction the formula is true for all positive integers n.

Solutionbank FP1
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Review Exercise
Exercise A, Question 46

Question:

$f(n) = (2n + 1)7^n - 1.$

Prove by induction that, for all positive integers n, $f(n)$ is divisible by 4.

Solution:

$f(n) = (2n+1)7^n - 1$

Let $n = 1$

$f(1) = (2\times 1 + 1)7^1 - 1 = 3\times 7 - 1 = 20$

20 is divisible by 4, so "f(n) is divisible by 4" is true for $n = 1$.

All inductions need to be shown to be true for a small number, usually 1. In this question, you need to check that when you substitute $n = 1$, the number you obtain is divisible exactly by 4.

Consider $f(k+1) - f(k)$

$$f(k+1) - f(k) = (2(k+1)+1)7^{k+1} - 1 - ((2k+1)7^k - 1)$$
$$= (2k+3)7^{k+1} - (2k+1)7^k$$
$$= (2k+3)7\times 7^k - (2k+1)7^k$$
$$= (14k+21)7^k - (2k+1)7^k$$
$$= (14k+21-2k-1)7^k$$
$$= (12k+20)7^k = 4(3k+5)7^k \ \ldots \ \ast$$

So 4 is a factor of $f(k+1) - f(k)$.

When you are trying to prove, by induction, that an expression $f(n)$ is divisible by a number, it is often a good start to try and show that $f(k+1) - f(k)$ is divisible by the same number. This does not always work but it is worth trying!

Showing that "$f(k+1) - f(k)$ has a factor of 4" is exactly the same thing as showing that "$f(k+1) - f(k)$ is divisible by 4".

Assume that $f(k)$ is divisible by 4.

It would follow that $f(k) = 4m$, where m is an integer.

From $\ast$

$$f(k+1) = f(k) + 4(3k+5)7^k$$
$$= 4m + 4(3k+5)7^k$$
$$= 4(m + (3k+5)7^k)$$

So $f(k+1)$ is divisible by 4.

The essential point here is that if both $f(k)$ and $4(3k+5)7^k$ are divisible by 4, then their sum, $f(k+1)$ is divisible by 4.

$f(n)$ is divisible by 4 for $n = 1$, and, if it is divisible by 4 for $n = k$, then it divisible by 4 for $n = k+1$.

By mathematical induction, $f(n)$ is divisible by 4 for all positive integers n.

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Review Exercise
Exercise A, Question 47

Question:

$\mathbf{A} = \begin{pmatrix} 1 & c \\ 0 & 2 \end{pmatrix}$, where c is a constant.

Prove by induction that, for all positive integers n,

$\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$

Solution:

$\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$

Let $n = 1$

$\mathbf{A}^1 = \begin{pmatrix} 1 & (2^1 - 1)c \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 2 \end{pmatrix}$

This is $\mathbf{A}$, as defined in the question, so the result is true for $n = 1$.

You need to begin by showing the result is true for $n = 1$. You substitute $n = 1$ into the printed expression for $\mathbf{A}^n$ and check that you get the matrix $\mathbf{A}$ as given in the question.

Assume the result is true for $n = k$.

That is $\mathbf{A}^k = \begin{pmatrix} 1 & (2^k - 1)c \\ 0 & 2^k \end{pmatrix}$ ... ... $*$

$$\begin{aligned}
\mathbf{A}^{k+1} &= \mathbf{A}^k.\mathbf{A} \\
&= \begin{pmatrix} 1 & (2^k - 1)c \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & c + 2(2^k - 1)c \\ 0 & 2.2^k \end{pmatrix} \\
&= \begin{pmatrix} 1 & c + 2^{k+1}c - 2c \\ 0 & 2^{k+1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2^{k+1}c - c \\ 0 & 2^{k+1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & (2^{k+1} - 1)c \\ 0 & 2^{k+1} \end{pmatrix}
\end{aligned}$$

Keep in mind as you multiply out the matrices that you are aiming at the expression $\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$ with each n replaced by $k+1$.

$2.2^k = 2^1.2^k = 2^{k+1}$ by one of the laws of indices. You use this twice.

This is the result obtained by substituting $n = k+1$ into the result $\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$ and so the result is true for $n = k+1$.

The result is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the result is true for all positive integers n.

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Review Exercise
Exercise A, Question 48

Question:

Given that $u_1 = 4$ and that $2u_{r+1} + u_r = 6$, use mathematical induction to prove that $u_n = 2 - \left(-\frac{1}{2}\right)^{n-2}$, for $n \varepsilon \mathbb{Z}^+$.

Solution:

$u_n = 2 - \left(-\frac{1}{2}\right)^{n-2}$

Let $n = 1$

$u_1 = 2 - \left(-\frac{1}{2}\right)^{1-2} = 2 - \left(-\frac{1}{2}\right)^{-1}$

$= 2 - (-2) = 4$

As the question gives $u_1 = 4$, the formula is true for $n = 1$.

All inductions need to be shown to be true for a small number, usually 1. In this question $u_1 = 4$ is part of the data of the question and you have to start by showing that $u_n = 4^n + 3n + 1$ satisfies $u_1 = 4$.

Using $a^{-1} = \frac{1}{a}$, $\left(-\frac{1}{2}\right)^{-1} = \frac{1}{-\frac{1}{2}} = 1 \times -\frac{2}{1} = -2$

Assume the formula is true for $n = k$.

That is $u_k = 2 - \left(-\frac{1}{2}\right)^{k-2}$ … … ✱

The **induction hypothesis** is just the formula you are asked to prove, with the ns replaced by ks.

$2u_{k+1} = 6 - u_k$

$2u_{k+1} = 6 - \left[2 - \left(-\frac{1}{2}\right)^{k-2}\right] = 4 + \left(-\frac{1}{2}\right)^{k-2}$, using ✱

The **induction hypothesis** allows you to substitute $2 - \left(-\frac{1}{2}\right)^{k-2}$ for u_k.

Hence, dividing both sides of the equation by 2

$u_{k+1} = 2 + \frac{1}{2}\left(-\frac{1}{2}\right)^{k-2}$

$= 2 - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^{k-2} = 2 - \left(-\frac{1}{2}\right)^{k-1}$

$\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^{k-2} = \left(-\frac{1}{2}\right)^{1}\left(-\frac{1}{2}\right)^{k-2}$

$= \left(-\frac{1}{2}\right)^{1+k-2} = \left(-\frac{1}{2}\right)^{k-1}$

This is the result obtained by substituting $n = k+1$ into the formula $u_n = 2 - \left(-\frac{1}{2}\right)^{n-2}$ and so the formula is true for $n = k+1$.

The formula is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the formula is true for all positive integers n, that is $n \in \mathbb{Z}^+$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 49

Question:

Prove by induction that, for all $n \in \mathbb{Z}^{+}$, $\sum_{r=1}^{n} r\left(\frac{1}{2}\right)^{r} = 2 - \left(\frac{1}{2}\right)^{n}(n+2)$.

Solution:

$$\sum_{r=1}^{n} r\left(\tfrac{1}{2}\right)^r = 2-\left(\tfrac{1}{2}\right)^n (n+2)$$

Let $n=1$.
The left-hand side becomes

$$\sum_{r=1}^{1} r\left(\tfrac{1}{2}\right)^r = 1\times\tfrac{1}{2} = \tfrac{1}{2}$$

$\sum_{r=1}^{1} r\left(\tfrac{1}{2}\right)^r$ consists of just one term. That is $r\left(\tfrac{1}{2}\right)^r$ with 1 substituted for r, which gives $\tfrac{1}{2}$.

The right-hand side becomes

$$2-\left(\tfrac{1}{2}\right)^1 (1+2) = 2-\tfrac{1}{2}\times 3 = \tfrac{1}{2}$$

The left-hand side and the right-hand side are equal and so the summation is true for $n=1$.

Assume the summation is true for $n=k$.

That is $\sum_{r=1}^{k} r\left(\tfrac{1}{2}\right)^r = 2-\left(\tfrac{1}{2}\right)^k (k+2)$ … … ∗

$$\sum_{r=1}^{k+1} r\left(\tfrac{1}{2}\right)^r = \sum_{r=1}^{k} r\left(\tfrac{1}{2}\right)^r 2+(k+1)\left(\tfrac{1}{2}\right)^{k+1}$$

$$= 2-\left(\tfrac{1}{2}\right)^k (k+2)+(k+1)\left(\tfrac{1}{2}\right)^{k+1}, \text{ using } *$$

$$= 2-\left(\tfrac{1}{2}\right)^{k+1} 2(k+2)+(k+1)\left(\tfrac{1}{2}\right)^{k+1}$$

$$= 2-\left(\tfrac{1}{2}\right)^{k+1}\left[2(k+2)-(k+1)\right]$$

$$= 2-\left(\tfrac{1}{2}\right)^{k+1}[k+3]$$

$$= 2-\left(\tfrac{1}{2}\right)^{k+1}((k+1)+2)$$

You are aiming at an expression where the n in $\left(\tfrac{1}{2}\right)^n$, on the right-hand side of the summation in the question, has been replaced by $k+1$. Replacing $\left(\tfrac{1}{2}\right)^k$ by the equal $\left(\tfrac{1}{2}\right)^{k+1}\times 2$ will give you $\left(\tfrac{1}{2}\right)^{k+1}$ as a common factor of the second and third terms.

This is the result obtained by substituting $n=k+1$ into the right-hand side of the summation and so the summation is true for $n=k+1$.

The summation is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the summation is true for all positive integers n, that is $n\in\mathbb{Z}^+$.

Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 50

Question:

$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$

Prove by induction that, for all positive integers *n*,

$\mathbf{A}^n = \begin{pmatrix} 2n+1 & n \\ -4n & -2n+1 \end{pmatrix}$

Solution:

$$\mathbf{A}^n = \begin{pmatrix} 2n+1 & n \\ -4n & -2n+1 \end{pmatrix}$$

Let $n=1$

$$\mathbf{A}^1 = \begin{pmatrix} 2+1 & 1 \\ -4 & -2+1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$$

This is $\mathbf{A}$, as defined in the question, so the result is true for $n=1$.

You need to begin by showing the result is true for $n=1$. You substitute $n=1$ into the printed expression for $\mathbf{A}^n$ and check that you get the matrix $\mathbf{A}$, as given in the question.

Assume the result is true for $n=k$.

That is $\mathbf{A}^k = \begin{pmatrix} 2k+1 & k \\ -4k & -2k+1 \end{pmatrix}$

The **induction hypothesis** is the result you are asked to prove with all the ns replaced by ks.

$$\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A}$$

$$= \begin{pmatrix} 2k+1 & k \\ -4k & -2k+1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3(2k+1)-4k & 2k+1-k \\ -12k-4(-2k+1) & -4k-(-2k+1) \end{pmatrix}$$

$$= \begin{pmatrix} 2k+3 & k+1 \\ -4k-4 & -2k+1 \end{pmatrix}$$

$$= \begin{pmatrix} 2(k+1)+1 & k+1 \\ -4(k+1) & -2(k+1)+1 \end{pmatrix}$$

$\mathbf{A}^{k+1}$ is the matrix $\mathbf{A}$, multiplied by itself k times, multiplied by $\mathbf{A}$ one more time. $\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A}^1 = \mathbf{A}^k.\mathbf{A}$. This is one of the index laws applied to matrices.

This is the result obtained by substituting $n=k+1$ into the result $\mathbf{A}^n = \begin{pmatrix} 2n+1 & n \\ -4n & -2n+1 \end{pmatrix}$ and so the result is true for $n=k+1$.

The result is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the result is true for all positive integers n.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 51

Question:

Given that $f(n) = 3^{4n} + 2^{4n+2}$,

a show that, for $k \varepsilon \mathbb{Z}^+$, $f(k+1) - f(k)$ is divisible by 15,

b prove that, for $n \varepsilon \mathbb{Z}^+$, $f(n)$ is divisible by 5.

Solution:

a $\mathrm{f}(n)=3^{4n}+2^{4n+2}$

$$\begin{aligned}
\mathrm{f}(k+1)-\mathrm{f}(k)&=3^{4(k+1)}+2^{4(k+1)+2}-\left(3^{4k}+2^{4k+2}\right)\\
&=3^{4k+4}-3^{4k}+2^{4k+6}-2^{4k+2}\\
&=3^{4k}\left(3^4-1\right)+2^{4k}\left(2^6-2^2\right)\\
&=3^{4k}\times 80+2^{4k}\times 60\\
&=3^{4k-1}\times 3\times 80+2^{4k}\times 60\\
&=240\times 3^{4k-1}+60\times 2^{4k}\\
&=15\left(16\times 3^{4k-1}+4\times 2^{4k}\right) \quad \ast
\end{aligned}$$

For all $k\in\mathbb{Z}^+$, $\left(16\times 3^{4k-1}+4\times 2^{4k}\right)$ is an integer, and, hence, $\mathrm{f}(k+1)-\mathrm{f}(k)$ is divisible by 15.

At this stage $\mathrm{f}(k+1)-\mathrm{f}(k)$ is clearly divisible by 10 (and 20) but to obtain that the expression is divisible by 15, you have to obtain a 3, to go with the 80, by writing 3^{4k} as $3^{4k-1}\times 3^1$.

This shows that 15 is a factor of $\mathrm{f}(k+1)-\mathrm{f}(k)$ and this is the equivalent to showing that $\mathrm{f}(k+1)-\mathrm{f}(k)$ is exactly divisible by 15. Note that the result would not be true for negative integers as, for example, 4×2^{4k} would be a fraction less than one.

b Let $n=1$

$\mathrm{f}(1)=3^4+2^6=81+64=145=5\times 29$

So $\mathrm{f}(n)$ is divisible by 5 for $n=1$.

Assume that $\mathrm{f}(k)$ is divisible by 5.

It would follow that $\mathrm{f}(k)=5m$, where m is an integer.

From $\ast$

$$\begin{aligned}
\mathrm{f}(k+1)&=\mathrm{f}(k)+15\left(16\times 3^{4k-1}+4\times 2^{4k}\right)\\
&=5m+15\left(16\times 3^{4k-1}+4\times 2^{4k}\right)\\
&=5\left(m+3\left(16\times 3^{4k-1}+4\times 2^{4k}\right)\right)
\end{aligned}$$

So $\mathrm{f}(k+1)$ is divisible by 5.

An expression which is divisible by 15 is certainly divisible by 5, which is all that is required for part (b).

If both $\mathrm{f}(k)$ and $15\left(16\times 3^{4k-1}+4\times 2^{4k}\right)$ are divisible by 5, then their sum, $\mathrm{f}(k+1)$ is divisible by 5.

$\mathrm{f}(n)$ is divisible by 5 for $n=1$, and, if it is divisible by 5 for $n=k$, then it divisible by 5 for $n=k+1$.

By mathematical induction, $\mathrm{f}(n)$ is divisible by 5 for all $n\in\mathbb{Z}^+$.

Although $\mathrm{f}(k+1)-\mathrm{f}(k)$ is divisible by 15, $\mathrm{f}(n)$ is never divisible by 15 for any n. After reading part (a), you might misread the question and attempt to prove that 15 was a factor of $\mathrm{f}(n)$. It is always necessary to read questions carefully.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 52

Question:

$f(n) = 24 \times 2^{4n} + 3^{4n}$, where n is a non-negative integer.

a Write down $f(n+1) - f(n)$.

b Prove, by induction, that $f(n)$ is divisible by 5.

Solution:

a $\mathrm{f}(n) = 24 \times 2^{4n} + 3^{4n}$

$\mathrm{f}(n+1) - \mathrm{f}(n) = 24 \times 2^{4(n+1)} + 3^{4(n+1)} - 24 \times 2^{4n} - 3^{4n}$

This is an acceptable answer for part (a). However, reading ahead, the question concerns divisibility by 5. So it is sensible to further work on this expression and show that it is divisible by 5.

b $\mathrm{f}(n+1) - \mathrm{f}(n)$

$= 24 \times 2^{4n+4} - 24 \times 2^{4n} + 3^{4n+4} - 3^{4n}$

$= 24 \times 2^{4n}(2^4 - 1) + 3^{4n}(3^4 - 1)$

$= 24 \times 2^{4n} \times 15 + 3^{4n} \times 80$

$= 5(72 \times 2^{4n} + 16 \times 3^{4n})$ ... ✱

In the middle of a question it is easy to forget that, in all inductions, you need to show that the result is true for a small number. This is usually 1 but this question asks you to show a result is true for all non-negative integers and 0 is a non-negative integer, so you should begin with 0.

Let $n = 0$

$\mathrm{f}(0) = 24 \times 2^0 + 3^0 = 24 + 1 = 25$

So $\mathrm{f}(n)$ is divisible by 5 for $n = 0$.

Assume that $\mathrm{f}(k)$ is divisible by 5.

It would follow that $\mathrm{f}(k) = 5m$, where m is an integer.

From ✱, substituting $n = k$ and rearranging,

$\mathrm{f}(k+1) = \mathrm{f}(k) + 5(72 \times 2^{4n} + 16 \times 3^{4n})$

$= 5m + 5(72 \times 2^{4n} + 16 \times 3^{4n})$

$= 5(m + 72 \times 2^{4n} + 16 \times 3^{4n})$

So $\mathrm{f}(k+1)$ is divisible by 5.

$\mathrm{f}(n)$ is divisible by 5 for $n = 0$, and, if it is divisible by 5 for $n = k$, then it divisible by 5 for $n = k+1$.

By mathematical induction, $\mathrm{f}(n)$ is divisible by 5 for all non-negative integers n.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 53

Question:

Prove that the expression $7^n + 4^n + 1$ is divisible by 6 for all positive integers n.

Solution:

Let $f(n) = 7^n + 4^n + 1$

If the question gives no label to the function, here $7^n + 4^n + 1$, it helps if you call it $f(n)$. You are going to have to refer to this function a number of times in your solution.

Let $n = 1$

$f(1) = 7^1 + 4^1 + 1 = 12$

12 is divisible by 6, so $f(n)$ is divisible by 6 for $n = 1$.

Consider $f(k+1) - f(k)$

This question gives you no hint to help you. With divisibility questions, it often helps to consider $f(k+1) - f(k)$ and try and show that this divides by the appropriate number, here 6. It does not always work and there are other methods which often work just as well or better. You should compare this question with questions **54** and **57** in this Review Exercise.

$$\begin{aligned} f(k+1) - f(k) &= 7^{k+1} + 4^{k+1} + 1 - (7^k + 4^k + 1) \\ &= 7^{k+1} - 7^k + 4^{k+1} - 4^k \\ &= 7^k(7-1) + 4^k(4-1) \\ &= 6 \times 7^k + 3 \times 4^k \\ &= 6 \times 7^k + 3 \times 4 \times 4^{k-1} \\ &= 6(7^k + 2 \times 4^{k-1}) \quad \ldots \ast \end{aligned}$$

So 6 is a factor of $f(k+1) - f(k)$.

Assume that $f(k)$ is divisible by 6.

It would follow that $f(k) = 6m$, where m is an integer.

From $\ast$

$$\begin{aligned} f(k+1) &= f(k) + 6(7^k + 2 \times 4^{k-1}) \\ &= 6m + 6(7^k + 2 \times 4^{k-1}) \\ &= 6(m + 7^k + 2 \times 4^{k-1}) \end{aligned}$$

If both $f(k)$ and $6(7^k + 2 \times 4^{k-1})$ are divisible by 6, then their sum, $f(k+1)$ is divisible by 6. You could write this down instead of the working shown here.

So $f(k+1)$ is divisible by 6.

$f(n)$ is divisible by 6 for $n = 1$, and, if it is divisible by 6 for $n = k$, then it divisible by 6 for $n = k+1$.

By mathematical induction, $f(n)$ is divisible by 6 for all positive integers n.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 54

Question:

Prove by induction that $4^n + 6n - 1$ is divisible by 9 for $n \varepsilon \mathbb{Z}^+$.

Solution:

Let $f(n) = 4^n + 6n - 1$

If the question gives no label to the function, here $4^n + 6n - 1$, it helps if you call it $f(n)$. You are going to have to refer to this function a number of times in your solution.

Let $n = 1$

$$f(1) = 4^1 + 6 - 1 = 9$$

So $f(n)$ is divisible by 9 for $n = 1$.

Assume that $f(k)$ is divisible by 9.

Then, for some integer m,

$f(k) = 4^k + 6k - 1 = 9m$

With divisibility questions, it often helps to consider $f(k+1) - f(k)$ and try and show that this divides by the appropriate number, here 9. This will work in this question. However the method shown here is, for this question, a neat one and you need to be aware of various alternative methods. No particular method works every time.

Rearranging

$$4^k = 9m - 6k + 1 \quad \ldots \ast$$

$$\begin{aligned} f(k+1) &= 4^{k+1} + 6(k+1) - 1 \\ &= 4 \times 4^k + 6k + 5 \\ &= 4 \times (9m - 6k + 1) + 6k + 5 \\ &= 36m - 24k + 4 + 6k + 5 = 36m - 18k + 9 \\ &= 9(4m - 2k + 1) \end{aligned}$$

Here you substitute the expression for 4^k in $\ast$ for the 4^k in your expression for $f(k+1)$.

This is divisible by 9.

$f(n)$ is divisible by 9 for $n = 1$, and, if it is divisible by 9 for $n = k$, then it divisible by 9 for $n = k + 1$.

By mathematical induction, $f(n)$ is divisible by 9 for all $n \in \mathbb{Z}^+$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 55

Question:

Prove that the expression $3^{4n-1} + 2^{4n-1} + 5$ is divisible by 10 for all positive integers n.

Solution:

Let $f(n) = 3^{4n-1} + 2^{4n-1} + 5$

Let $n = 1$

$f(1) = 3^3 + 2^3 + 5 = 27 + 8 + 5 = 40 = 10 \times 4$

So $f(n)$ is divisible by 10 for $n = 1$.

Consider $f(k+1) - f(k)$

$$\begin{aligned} f(k+1) - f(k) &= 3^{4k+3} + 2^{4k+3} - 5 - \left(3^{4k-1} + 2^{4k-1} - 5\right) \\ &= 3^{4k+3} - 3^{4k-1} + 2^{4k+3} - 2^{4k-1} \\ &= 3^{4k-1}\left(3^4 - 1\right) + 2^{4k-3}\left(2^6 - 2^2\right) \\ &= 3^{4k-1} \times 80 + 2^{4k-3} \times 30 \\ &= 10\left(8 \times 3^{4k-1} + 3 \times 2^{4k-3}\right) \ \ldots * \end{aligned}$$

When you replace n by $k+1$ in, for example, 3^{4n-1} you get $3^{4(k+1)-1} = 3^{4k+4-1} = 3^{4k+3}$.

The index manipulation is quite complicated here. For example, $2^{4k-3} \times 2^6 = 2^{4k-3+6} = 2^{4k+3}$.

Assume that $f(k)$ is divisible by 10.

It would follow that $f(k) = 10m$, where m is an integer.

From *

$$\begin{aligned} f(k+1) &= f(k) + 10\left(8 \times 3^{4k-1} + 3 \times 2^{4k-3}\right) \\ &= 10m + 10\left(8 \times 3^{4k-1} + 3 \times 2^{4k-3}\right) \\ &= 10\left(m + \left(8 \times 3^{4k-1} + 3 \times 2^{4k-3}\right)\right) \end{aligned}$$

If both $f(k)$ and $10\left(8 \times 3^{4k-1} + 3 \times 2^{4k-3}\right)$ are divisible by 10, then their sum, $f(k+1)$ is divisible by 10. If you preferred, you could write this down instead of the working shown here.

So $f(k+1)$ is divisible by 10.

$f(n)$ is divisible by 10 for $n = 1$, and, if it is divisible by 10 for $n = k$, then it divisible by 10 for $n = k+1$.

By mathematical induction, $f(n)$ is divisible by 10 for all positive integers n.

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Review Exercise
Exercise A, Question 56

Question:

a Express $\frac{6x+10}{x+3}$ in the form $p+\frac{q}{x+3}$, where p and q are integers to be found.

The sequence of real numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 = 5.2$ and $u_{n+1} = \frac{6u_n+10}{u_n+3}$.

b Prove by induction that $u_n > 5$, for $n \varepsilon \mathbb{Z}^+$.

Solution:

a
$$\frac{6x+10}{x+3}=\frac{6x+18-8}{x+3}=\frac{6(x+3)-8}{x+3}$$
$$=\frac{6(x+3)}{x+3}-\frac{8}{x+3}=6-\frac{8}{x+3}$$
$p=6,\ q=-8$

You may use any correct method to carry out the division in part (a). Methods can be found in Chapter 1 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 2.

b
$u_1 = 5.2 > 5$
So $u_n > 5$ for $n = 1$.

It is obvious that $5.2 > 5$ but all inductions need to be shown to be true for a small number, usually 1, and you must remember to write down that $5.2 > 5$ shows that the result is true for $n = 1$.

Assume that $u_k > 5$
If $u_k > 5$, there exists a positive number ε such that $u_k = 5+\varepsilon$.

$$u_{k+1}=\frac{6u_k+10}{u_k+3}=6-\frac{8}{u_k+3}, \text{ using the result in part (a)}$$
$$=6-\frac{8}{5+\varepsilon+3}=6-\frac{8}{8+\varepsilon}$$
$$>6-1=5$$

If $\varepsilon > 0$ then $\frac{8}{8+\varepsilon}$ is less than one – the numerator is smaller than the denominator. It follows that $6-\frac{8}{8+\varepsilon}$ will be bigger than 5.

So $u_{k+1} > 5$
$u_1 > 5$ and, if $u_k > 5$, then $u_{k+1} > 5$.
By mathematical induction, $u_n > 5$ for all $n \in \mathbb{Z}^+$.

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Review Exercise
Exercise A, Question 57

Question:

Given that $n \varepsilon \mathbb{Z}^+$, prove, by mathematical induction, that $2(4^{2n+1}) + 3^{3n+1}$ is divisible by 11.

Solution:

Let $\text{f}(n) = 2(4^{2n+1}) + 3^{3n+1}$

Let $n = 1$

$\text{f}(1) = 2(4^{2+1}) + 3^{3+1} = 2 \times 4^3 + 3^4$

$\quad = 2 \times 64 + 81 = 209 = 11 \times 19$

So $\text{f}(n)$ is divisible by 11 for $n = 1$.

Assume that $\text{f}(k)$ is divisible by 11.

Then, for some integer m,

$\text{f}(k) = 2(4^{2k+1}) + 3^{3k+1} = 11m$

Rearranging

$\quad 2(4^{2k+1}) = 11m - 3^{3k+1} \quad \ldots \ast$

The method of considering $\text{f}(k+1) - \text{f}(k)$ is very difficult to make work in this question and this alternative method is easier here.

$\text{f}(k+1) = 2(4^{2k+3}) + 3^{3k+4}$

$\quad = 2(4^{2k+1} \times 4^2) + 3^{3k+4}$

$\quad = 16 \times 2(4^{2k+1}) + 3^{3k+4}$

$\quad = 16 \times (11m - 3^{3k+1}) + 3^{3k+4}$

$\quad = 176m - 16 \times 3^{3k+1} + 27 \times 3^{3k+1}$

$\quad = 176m + 11 \times 3^{3k+1}$

$\quad = 11(16m + 3^{3k+1})$

This is divisible by 11.

Here you substitute the expression for $2(4^{2k+1})$ in $\ast$ for the $2(4^{2k+1})$ in your expression for $\text{f}(k+1)$.

$3^{3k+4} = 3^{3k+1+3} = 3^{3k+1} \times 3^3 = 27 \times 3^{3k+1}$

This is needed to get a common factor of 3^{3k+1} in the second and third terms of the expression.

$\text{f}(n)$ is divisible by 11 for $n = 1$, and, if it is divisible by 11 for $n = k$, then it is divisible by 11 for $n = k+1$.

By mathematical induction, $\text{f}(n)$ is divisible by 11 for all $n \in \mathbb{Z}^+$.